Global solvability of the free-boundary problem for one-dimensional motion of a self-gravitating viscous radiative and reactive gas

By Morimichi UMEHARA and Atusi TANI

Department of Mathematics, Keio University, 3-4-1 Hiyoshi, Yokohama 223-8522, Japan

(Communicated by Heisuke HIRONAKA, M.J.A., June 10, 2008)

Abstract: In this paper we consider a system of equations describing the one-dimensional motion of a self-gravitating, radiative and chemically reactive gas having the free-boundary. For arbitrary large, smooth initial data we prove the unique existence, global in time, of a classical solution of the corresponding problem with fixed domain, obtained by the Lagrangian mass transformation.

Key words: Global solution; free-boundary problem; self-gravitation; radiative gas; reactive gas; Lagrangian mass coordinate.

1. Introduction. We consider the free-boundary problem describing the motion of a compressible, viscous and heat-conductive gas. We take into account a more phenomenal situation: The gas is self-gravitating, radiative and chemically reactive. In [12], such a problem was formulated in spacially one-dimensional case and solved globally in time for arbitrary large, smooth initial data under some (physically correct) hypotheses on the thermal conductivity and so on. In this paper we improve the result in [12] in respect to the restriction guaranteeing the global in time solvability of the problem.

The gaseous motion mentioned above, especially in the processes of the unimolecular reactions whose kinetic order is one, is described by the following four equations in the Lagragian mass coordinate:

$$(1.1)\qquad \begin{cases} v_t = u_x, \\ u_t = \left(-p + \mu \dfrac{u_x}{v}\right)_x - G\left(x - \dfrac{1}{2}\right), \\ e_t = \left(-p + \mu \dfrac{u_x}{v}\right) u_x + \left(\kappa \dfrac{\theta_x}{v}\right)_x + \lambda \phi z, \\ z_t = d\left(\dfrac{z_x}{v^2}\right)_x - \phi z \end{cases}$$

in $\Omega \times (0, \infty)$ with $\Omega := (0, 1)$. Here the specific volume $v = v(x, t)$, the velocity $u = u(x, t)$, the absolute temperature $\theta = \theta(x, t)$ and the mass fraction of the reactant $z = z(x, t)$ are the unknown quantities, and positive constants μ, G, d and λ are the bulk viscosity, the Newtonian gravitational constant, the species diffusion coefficient and the difference in heat between the reactant and the product, respectively. The pressure p and the internal energy per unit mass e are defined by

$$(1.2)\qquad \begin{cases} p = p(v, \theta) = R\dfrac{\theta}{v} + \dfrac{a}{3}\theta^4, \\ e = e(v, \theta) = c_{\mathrm{v}}\theta + av\theta^4, \end{cases}$$

where positive constants R, c_{v} and a are the perfect gas constant, the specific heat capacity at constant volume and the radiation-density constant, respectively. Second terms in the right-hand sides of both relations in (1.2) represent the effect of radiation phenomena, whose forms are given by the famous Stefan-Boltzmann law. In the radiating regime, it is natural to take into account the heat flux from the radiative contribution, not only from the heat-conductive contribution. As such a simple one (see [13]), we assume that the thermal conductivity $\kappa = \kappa(v, \theta)$ has the form

$$(1.3)\qquad \kappa(v, \theta) = \kappa_1 + \kappa_2 v\theta^q$$

with positive constants κ_1, κ_2 and q. Furthermore we assume that the reaction rate function $\phi = \phi(\theta)$ is defined, from the Arrhenius law, by

$$(1.4)\qquad \phi(\theta) = K\theta^\beta \mathrm{e}^{-A/\theta},$$

where positive constants K and A are the coefficient of rate of the reactant and the activation energy,

2000 Mathematics Subject Classification. Primary 35Q30, 35R30, 35R76, 35N10.

respectively, and β is a non-negative real number.

Imposed boundary condition is for $t > 0$

$$(1.5)\qquad \left(-R\frac{\theta}{v} + \mu\frac{u_x}{v}, \theta_x, z_x\right)\Big|_{x=0,1} = (-p_e, 0, 0)$$

with the external pressure p_e (a positive constant), and the initial condition is for $x \in \overline{\Omega}$

$$(1.6)\qquad (v, u, \theta, z)|_{t=0} = (v_0(x), u_0(x), \theta_0(x), z_0(x)).$$

We give some more remarks on our problem. The external force appeared in the last term of the right-hand side of $(1.1)^2$ is given as follow: First we assume that the external force per unit mass $f = f(x,t)$ in the Lagrangian mass coordinate is given by $f = -U_x/v$, where $U = U(x,t)$ is the solution of the boundary value problem for each $t > 0$

$$(1.7)\qquad \begin{cases} \left(\dfrac{U_x}{v}\right)_x = G & \text{in } \Omega, \\ U|_{x=0} = U|_{x=1} = 0. \end{cases}$$

One can regard that f defined above is the one-dimensional "self-gravitation" (U is the corresponding "self-gravitational potential"), similar to the general three-dimensional one. From (1.7) we can derive the explicit formula of $f(x,t)$,

$$(1.8)\qquad f(x,t) = -G\left(x - \frac{\int_0^1 \xi v(\xi,t)\,\mathrm{d}\xi}{\int_0^1 v(\xi,t)\,\mathrm{d}\xi}\right).$$

Here, if $(\hat{v}, \hat{u}, \hat{\theta}, \hat{z})$ satisfies

$$u_t = \left(-p + \mu\frac{u_x}{v}\right)_x + f$$

instead of $(1.1)^2$, and $(1.1)^1$, $(1.1)^3$, $(1.1)^4$, (1.5), then $(v, u, \theta, z) = (\hat{v}, \hat{u} - \int_0^1 \hat{u}\,\mathrm{d}x, \hat{\theta}, \hat{z})$ satisfies (1.1) and (1.5). In addition, integrating $(1.1)^2$ over $\overline{\Omega} \times [0,t]$ under (1.5) gives

$$\int_0^1 u(x,t)\,\mathrm{d}x = \int_0^1 u(x,0)\,\mathrm{d}x,$$

whose left-hand side is identically equal to zero when $u = \hat{u} - \int_0^1 \hat{u}\,\mathrm{d}x$. Hence, it is natural for us to consider the system (1.1)–(1.6) under the condition

$$(1.9)\qquad \int_0^1 u_0(x)\,\mathrm{d}x = 0.$$

We shall introduce some function spaces used in this paper (see, for detail, [6]). Let m be a non-negative integer and $0 < \sigma$, $\sigma' < 1$. By $C^{m+\sigma}(\Omega)$ we denote the spaces of functions $u = u(x)$ which has bounded derivatives up to order m and $\mathrm{d}^m u/\mathrm{d}x^m$ is uniformly Hölder continuous with exponent σ. Let T be a positive constant and $Q_T := \Omega \times (0,T)$. For a function u defined on Q_T we say that $u \in C_{x,t}^{\sigma,\sigma'}(Q_T)$ if

$$|u|^{(0)} := \sup_{(x,t)\in Q_T} |u(x,t)| < \infty$$

and u is uniformly Hölder continuous in x and t with exponent σ and σ', respectively. Its norm is denoted by $|\cdot|_{\sigma,\sigma'}$. We also say that $u \in C_{x,t}^{2+\sigma,1+\sigma/2}(Q_T)$ if u is bounded, has bounded derivative u_x, and $(u_{xx}, u_t) \in (C_{x,t}^{\sigma,\sigma/2}(Q_T))^2$. Its norm is denoted by $|\cdot|_{2+\sigma,1+\sigma/2}$.

Our main result is

Theorem 1 (Global Solution)**.** *Let $\alpha \in (0,1)$, $q \geq 3$ and $0 \leq \beta < q + 9$. Assume that*

$$(v_0, u_0, \theta_0, z_0) \in C^{1+\alpha}(\Omega) \times (C^{2+\alpha}(\Omega))^3$$

satisfies corresponding compatibility conditions, (1.9) and $v_0(x) > 0$, $\theta_0(x) > 0$, $0 \leq z_0(x) \leq 1$ for any $x \in \overline{\Omega}$. Then there exists a unique solution (v, u, θ, z) of the initial-boundary value problem (1.1)–(1.6) such that for any positive number T

$$(v,\, v_x, v_t) \in (C_{x,t}^{\alpha,\,\alpha/2}(Q_T))^3,$$
$$(u, \theta, z) \in (C_{x,t}^{2+\alpha,\,1+\alpha/2}(Q_T))^3.$$

Moreover for any $(x,t) \in \overline{Q_T}$

$$v(x,t) > 0, \quad \theta(x,t) > 0, \quad 0 \leq z(x,t) \leq 1.$$

The global in time solvability of this problem in the same function spaces as in Theorem 1 was already established in [12] for $4 \leq q \leq 16$ and $0 \leq \beta \leq 13/2$. Theorem 1 is its improved version. We note that an analogous result was also reported in [13] for a three-dimensional spherically symmetric gaseous model having a central rigid core.

We mention some related results concerning our problem, briefly. For compressible, viscous and heat-conductive model in one space dimension, many studies have been done including the case for large, smooth initial data e.g., Kazhikhov-Shelukhin [5], Kazhikhov [4] and Nagasawa [7]. Ducomet [1] and Ducomet-Zlotnik [2,3] studied one-dimensional gaseous models rather similar to ours, i.e., radiative and reactive one with the free-boundary in the external force field. In [1] the temporally global existence of the solution was shown for $q = 4$ in (1.3) and $\beta = 0$ in (1.4). However, in a series of papers [1–3] they adopted as a self-gravitation, a special form independent of time variable explicitly in the Lagrangian mass coordinate system, not the exact form (1.8). That is called the "pancakes model" relevant to some large-scale structure of the universe (see [9]). In addition, although the temporally global existence of the solution for any $q \geq 2$ was established recently in [2,3], they were discussed not for the pure free-boundary case (1.5) but for the one with partially Dirichlet boundary condition, i.e., $\theta|_{x=0}$ (or $\theta|_{x=1}$) $= \theta_\Gamma$ with positive constant θ_Γ.

In order to prove Theorem 1, after historical results about local existence theorem for general compressible, viscous and heat-conductive fluids [8,10,11], it is sufficient to establish the following *a priori* boundedness.

Proposition 1 (*A priori* Estimates)**.** *Let T be an arbitrary positive number. Assume that α, q, β and the initial data satisfy the hypotheses of Theorem* 1*, and that the problem* (1.1)–(1.6) *has a solution (v, u, θ, z) such that*

$$(v,\, v_x, v_t) \in (C^{\alpha,\,\alpha/2}_{x,t}(Q_T))^3,$$
$$(u, \theta, z) \in (C^{2+\alpha,\,1+\alpha/2}_{x,t}(Q_T))^3.$$

Then there exists a positive constant C depending on the initial data and T such that

$$|v,\, v_x, v_t|_{\alpha,\,\alpha/2},\ |u, \theta, z|_{2+\alpha,\,1+\alpha/2} \leq C$$

and for any $(x,t) \in \overline{Q_T}$

$$v(x,t),\ \theta(x,t) \geq 1/C, \quad 0 \leq z(x,t) \leq 1.$$

2. Sketch of proof of Proposition 1. In proving Proposition 1, we need several lemmas concerning the estimates of the solution and its derivatives. We use C_0 and C, C_T as positive constants depending on the initial data and other constants, but the former does not depend on T, and $\|\cdot\|$ as the usual $L^2(\Omega)$-norm.

In [12] we already obtained the following estimates of the solution.

Lemma 1. *For any $t \in [0, T]$*

$$\int_0^1 \left(\frac{1}{2}u^2 + e + \lambda z + f(x)v\right) \mathrm{d}x = E_0,$$

$$U(t) + \int_0^t V(\tau)\,\mathrm{d}\tau \leq C_0,$$

$$\int_0^1 z\,\mathrm{d}x + \int_0^t \int_0^1 \phi z\,\mathrm{d}x\,\mathrm{d}\tau = \int_0^1 z_0\,\mathrm{d}x,$$

$$\int_0^1 \frac{1}{2} z^2\,\mathrm{d}x + \int_0^t \int_0^1 \left(\frac{d}{v^2} {z_x}^2 + \phi z^2\right) \mathrm{d}x\,\mathrm{d}\tau = \int_0^1 \frac{1}{2} {z_0}^2\,\mathrm{d}x,$$

$$\int_0^t \max_{x \in \overline{\Omega}} \theta(x,\tau)^\gamma\,\mathrm{d}\tau \leq C_T$$

for $0 \leq \gamma \leq q+4$ $(q \geq 0)$,

$$\int_0^t \|u_x\|^2\,\mathrm{d}\tau \leq C_T,$$

$$\|v_x\|^2 + \int_0^t \int_0^1 \theta {v_x}^2\,\mathrm{d}x\,\mathrm{d}\tau \leq C_T \quad \textit{if } q \geq 2,$$

$$\int_0^t \|u_x\|^3_{L^3(\Omega)}\,\mathrm{d}\tau \leq C_T \quad \textit{if } q \geq 4, \tag{2.1}$$

and for any $(x,t) \in \overline{Q_T}$

$$0 \leq z(x,t) \leq 1,$$
$${C_T}^{-1} \leq v(x,t) \leq C_T. \tag{2.2}$$

Here

$$\begin{cases} E_0 := \displaystyle\int_0^1 \left(\frac{1}{2}{u_0}^2 + e_0 + \lambda z_0 + f(x)v_0\right) \mathrm{d}x, \\ U(t) := \displaystyle\int_0^1 [c_{\mathrm{v}}(\theta - 1 - \log\theta) + R(v - 1 - \log v)]\,\mathrm{d}x, \\ V(t) := \displaystyle\int_0^1 \left(\frac{\mu {u_x}^2}{v\theta} + \frac{\kappa {\theta_x}^2}{v\theta^2} + \lambda\frac{\phi}{\theta} z\right) \mathrm{d}x \end{cases}$$

and $e_0 := c_{\mathrm{v}}\theta_0 + a v_0 {\theta_0}^4$, $f(x) := p_e + \dfrac{1}{2}Gx(1-x)$.

For the proof of this lemma we mainly used the standard energy methods. Among them most important is the pointwise estimate (2.2) of v both from above and from below. This is derived from the representation formula of v originally due to Kazhikhov-Shelukhin [5]:

Lemma 2. *The identity*

$$v(x,t) = \frac{1}{\mathrm{P}(x,t)\mathrm{Q}(x,t)\mathrm{R}(x,t)} \times \left(v_0(x) + \frac{R}{\mu}\int_0^t \theta(x,\tau)\mathrm{P}(x,\tau)\mathrm{Q}(x,\tau)\mathrm{R}(x,\tau)\,\mathrm{d}\tau\right)$$

holds, where

$$\begin{cases} \mathrm{P}(x,t) := \exp\left[\dfrac{1}{\mu}\displaystyle\int_0^x (u_0(\xi) - u(\xi,t))\,\mathrm{d}\xi\right], \\ \mathrm{Q}(x,t) := \exp\left(\dfrac{1}{\mu} f(x)\,t\right), \\ \mathrm{R}(x,t) := \exp\left(-\dfrac{a}{3\mu}\displaystyle\int_0^t \theta(x,\tau)^4\,\mathrm{d}\tau\right). \end{cases}$$

Next, to obtain higher order estimates we define the function

$$\mathrm{K} = \mathrm{K}(v,\theta) := \int_0^\theta \frac{\kappa(v,\xi)}{v}\,\mathrm{d}\xi.$$

Since $(1.1)^3$ is rewritten as

$$(2.3)\qquad e_\theta\theta_t + \theta p_\theta u_x = \frac{\mu}{v}{u_x}^2 + \left(\frac{\kappa}{v}\theta_x\right)_x + \lambda\phi z,$$

multiplying this by K_t and integrating it over $\overline{\Omega}\times[0,t]$ yield

$$(2.4)\qquad \int_0^t\int_0^1 e_\theta\theta_t\mathrm{K}_t\,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t\int_0^1 \frac{\kappa}{v}\theta_x\mathrm{K}_{xt}\,\mathrm{d}x\,\mathrm{d}\tau = \int_0^t\int_0^1\left(-\theta p_\theta u_x + \frac{\mu}{v}{u_x}^2 + \lambda\phi z\right)\mathrm{K}_t\,\mathrm{d}x\,\mathrm{d}\tau.$$

Here

$$\begin{cases} \mathrm{K}_t = \dfrac{\kappa}{v}\theta_t + \mathrm{K}_v u_x, \\ \mathrm{K}_{xt} = \left(\dfrac{\kappa}{v}\theta_x\right)_t + \mathrm{K}_v u_{xx} + \mathrm{K}_{vv} v_x u_x + \left(\dfrac{\kappa}{v}\right)_v v_x\theta_t, \\ |\mathrm{K}_v|,\ |\mathrm{K}_{vv}| \le C\theta. \end{cases}$$

Let us introduce the following quantities:

$$\begin{cases} X := \displaystyle\int_0^t\int_0^1 \bigl(1+\theta^{q+3}\bigr){\theta_t}^2\,\mathrm{d}x\,\mathrm{d}\tau, \\ Y := \displaystyle\max_{t\in[0,T]}\int_0^1\bigl(1+\theta^{2q}\bigr){\theta_x}^2\,\mathrm{d}x, \\ Z := \displaystyle\max_{t\in[0,T]}\|u_{xx}\|^2. \end{cases}$$

Then it is easily seen that by virtue of Cauchy-Schwarz' and standard interpolation inequalities we have

$$(2.5)\qquad \begin{cases} |\theta|^{(0)} \le C + CY^{\frac{1}{2q+6}}, \\ \displaystyle\max_{t\in[0,T]}\|u_x\|^2 \le C + CZ^{1/2}, \\ |u_x|^{(0)} \le C + CZ^{3/8}. \end{cases}$$

Estimating each term in (2.4), we can obtain

Lemma 3. *If $q\ge 2$ and $0\le\beta<q+9$, then there exists a number δ, $0<\delta<1$ such that*

$$(2.6)\qquad X + Y \le C_T\bigl(1+Z^\delta\bigr).$$

Proof. Let $q\ge 2$ and $\beta\ge 0$ first. Hereafter we use C_ε as a positive constant depending on $\varepsilon>0$. From the definition of X and Y one can immediately derive the inequalities

$$(2.7)\qquad \int_0^t\int_0^1 e_\theta\theta_t\cdot\frac{\kappa}{v}\theta_t\,\mathrm{d}x\,\mathrm{d}\tau \ge CX,$$

$$(2.8)\qquad \begin{aligned} &\int_0^t\int_0^1\frac{\kappa}{v}\theta_x\cdot\left(\frac{\kappa}{v}\theta_x\right)_t\mathrm{d}x\,\mathrm{d}\tau \\ &= \frac12\int_0^1\left(\frac{\kappa}{v}\theta_x\right)^2\mathrm{d}x - \frac12\int_0^1\left(\frac{\kappa_0}{v_0}{\theta_0}'\right)^2\mathrm{d}x \\ &\ge CY - C \end{aligned}$$

with $\kappa_0 := \kappa_1 + \kappa_2 v_0{\theta_0}^q$. In the same way as that in [12] it is not diffficult to get the following estimates.

$$(2.9)\qquad \begin{aligned} &\left|\int_0^t\int_0^1 e_\theta\theta_t\cdot\mathrm{K}_v u_x\,\mathrm{d}x\,\mathrm{d}\tau\right| \le \varepsilon X \\ &+ C_\varepsilon|{u_x}^2|^{(0)}\int_0^t\max_{x\in\overline{\Omega}}(1+\theta)^{1-q}\int_0^1(1+\theta)^4\,\mathrm{d}x\,\mathrm{d}\tau \\ &\le \varepsilon X + C_\varepsilon\Bigl(1+Z^{3/4}\Bigr); \end{aligned}$$

$$(2.10)\qquad \begin{aligned} &\left|\int_0^t\int_0^1\frac{\kappa}{v}\theta_x\cdot\mathrm{K}_v u_{xx}\,\mathrm{d}x\,\mathrm{d}\tau\right| \\ &\le C|1+\theta^{\frac{q}{2}+2}|^{(0)}\max_{t\in[0,T]}\|u_{xx}\|\int_0^t(1+V(\tau))\,\mathrm{d}\tau \\ &\le C + CY^{\frac{q/2+2}{2q+6}}Z^{1/2} \le \varepsilon Y + C_\varepsilon\Bigl(1+Z^{3/4}\Bigr); \end{aligned}$$

$$(2.11)\qquad \begin{aligned} &\left|\int_0^t\int_0^1\frac{\kappa}{v}\theta_x\cdot\mathrm{K}_{vv}v_x u_x\,\mathrm{d}x\,\mathrm{d}\tau\right| \\ &\le C\,|u_x|^{(0)}Y^{1/2}\left(\int_0^t\max_{x\in\overline{\Omega}}(1+\theta^2)\|v_x\|^2\,\mathrm{d}\tau\right)^{1/2} \\ &\le \varepsilon Y + C_\varepsilon\Bigl(1+Z^{3/4}\Bigr); \end{aligned}$$

$$(2.12)\qquad \begin{aligned} &\left|\int_0^t\int_0^1\theta p_\theta u_x\cdot\frac{\kappa}{v}\theta_t\,\mathrm{d}x\,\mathrm{d}\tau\right| \\ &\le \varepsilon X + C_\varepsilon|1+\theta^{q+5}|^{(0)}\int_0^t\|u_x\|^2\,\mathrm{d}\tau \\ &\le \varepsilon(X+Y) + C_\varepsilon; \end{aligned}$$

$$(2.13)\qquad \begin{aligned} &\left|\int_0^t\int_0^1\theta p_\theta u_x\cdot\mathrm{K}_v u_x\,\mathrm{d}x\,\mathrm{d}\tau\right| \\ &\le C\,|1+\theta^5|^{(0)}\int_0^t\|u_x\|^2\,\mathrm{d}\tau \le \varepsilon Y + C_\varepsilon; \end{aligned}$$

$$(2.14)\qquad \begin{aligned} &\left|\int_0^t\int_0^1\frac{\mu}{v}{u_x}^2\cdot\mathrm{K}_v u_x\,\mathrm{d}x\,\mathrm{d}\tau\right| \\ &\le C\,|u_x|^{(0)}\max_{t\in[0,T]}\|u_x\|^2\int_0^t\max_{x\in\overline{\Omega}}\theta\,\mathrm{d}\tau \le C\Bigl(1+Z^{7/8}\Bigr); \end{aligned}$$

$$(2.15)\qquad \begin{aligned} &\left|\int_0^t\int_0^1\lambda\phi z\cdot\mathrm{K}_v u_x\,\mathrm{d}x\,\mathrm{d}\tau\right| \\ &\le C\,|\theta u_x|^{(0)}\int_0^t\int_0^1\phi z\,\mathrm{d}x\,\mathrm{d}\tau \le \varepsilon Y + C_\varepsilon\Bigl(1+Z^{3/4}\Bigr). \end{aligned}$$

Next we have

$$(2.16)\quad \left|\int_0^t\int_0^1 \frac{\kappa}{v}\theta_x\cdot\left(\frac{\kappa}{v}\right)_v v_x\theta_t\,\mathrm{d}x\,\mathrm{d}\tau\right| \le \varepsilon X + C_\varepsilon\int_0^t \max_{x\in\overline{\Omega}}\left(\frac{\kappa}{v}\theta_x\right)^2\int_0^1\frac{1}{(1+\theta)^{q+3}}{v_x}^2\,\mathrm{d}x\,\mathrm{d}\tau.$$

We shall estimate the right-hand side of (2.16). At first we derive for any $t\in[0,T]$

$$\begin{aligned}(2.17)\quad \max_{x\in\overline{\Omega}}\left(\frac{\kappa}{v}\theta_x\right)^2(x,t) &\le \int_0^1\left(\frac{\kappa}{v}\theta_x\right)^2\mathrm{d}x + 2\int_0^1\left|\frac{\kappa}{v}\theta_x\right|\left|\left(\frac{\kappa}{v}\theta_x\right)_x\right|\mathrm{d}x\\ &\le C|1+\theta^{q+2}|^{(0)}V(t)+\\ &\quad + CV(t)^{1/2}\left[\int_0^1(1+\theta^{q+2})\left(\frac{\kappa}{v}\theta_x\right)_x^2\mathrm{d}x\right]^{1/2}.\end{aligned}$$

Substitute the inequality

$$\left(\frac{\kappa}{v}\theta_x\right)_x^2 \le C\Big({e_\theta}^2{\theta_t}^2+\theta^2{p_\theta}^2{u_x}^2+{u_x}^4+\phi^2z^2\Big),$$

which follows from (2.3), into the integrand of the rightmost hand side of (2.17). Then, noting the inequalities

$$\int_0^t\int_0^1(1+\theta^{q+2}){e_\theta}^2{\theta_t}^2\,\mathrm{d}x\,\mathrm{d}\tau \le C\,|1+\theta^5|^{(0)}X \le CX + CXY^{\frac{5}{2q+6}},$$

$$\begin{aligned}\int_0^t\int_0^1(1+\theta^{q+2})\theta^2{p_\theta}^2{u_x}^2\,\mathrm{d}x\,\mathrm{d}\tau &\le C\,|(1+\theta^2){u_x}^2|^{(0)}\int_0^t\max_{x\in\overline{\Omega}}(1+\theta^{q+4})\times\int_0^1(1+\theta^4)\,\mathrm{d}x\,\mathrm{d}\tau\\ &\le C + CY^{\frac{1}{q+3}} + CY^{\frac{1}{q+3}}Z^{3/4} + CZ^{3/4},\end{aligned}$$

$$\begin{aligned}\int_0^t\int_0^1(1+\theta^{q+2}){u_x}^4\,\mathrm{d}x\,\mathrm{d}\tau &\le |(1+\theta^{q+2}){u_x}^2|^{(0)}\int_0^t\|u_x\|^2\,\mathrm{d}\tau\\ &\le C + CY^{\frac{q+2}{2q+6}} + CY^{\frac{q+2}{2q+6}}Z^{3/4} + CZ^{3/4},\end{aligned}$$

$$\begin{aligned}&\int_0^t\int_0^1(1+\theta^{q+2})\phi^2z^2\,\mathrm{d}x\,\mathrm{d}\tau\\ &\le C\,|1+\theta^{q+2+\beta}|^{(0)}\int_0^t\int_0^1\phi z^2\,\mathrm{d}x\,\mathrm{d}\tau \le C + CY^{\frac{q+2+\beta}{2q+6}},\end{aligned}$$

we have, by integrating (2.17), for any $t\in[0,T]$

$$\begin{aligned}\int_0^t\max_{x\in\overline{\Omega}}\left(\frac{\kappa}{v}\theta_x\right)^2(x,\tau)\,\mathrm{d}\tau &\le C\Big(1+X^{1/2}+Y^{\frac{q/2+1+\beta/2}{2q+6}}\\ &\quad + Z^{3/8} + X^{1/2}Y^{\frac{5/2}{2q+6}} + Y^{\frac{q/2+1}{2q+6}}Z^{3/8}\Big)\\ &\le \varepsilon(X+Y) + C_\varepsilon\Big(1+Z^{3/4}\Big)\end{aligned}$$

for $0\le\beta<3q+10$. From this, the right-hand side of (2.16) is estimated from above by

$$(2.18)\qquad \varepsilon(X+Y)+C_\varepsilon\Big(1+Z^{3/4}\Big)$$

only for $0\le\beta<3q+10$. Moreover, the stronger restriction on β, $0\le\beta<q+9$, is necessary to get

$$\begin{aligned}(2.19)\quad \left|\int_0^t\int_0^1\lambda\phi z\cdot\frac{\kappa}{v}\theta_t\,\mathrm{d}x\,\mathrm{d}\tau\right| &\le \varepsilon X + C_\varepsilon|(1+\theta)^{q-3+\beta}|^{(0)}\int_0^t\int_0^1\phi z^2\,\mathrm{d}x\,\mathrm{d}\tau\\ &\le \varepsilon(X+Y)+C_\varepsilon.\end{aligned}$$

The estimate improved essentially from the one in [12] is

$$\left|\int_0^t\int_0^1\frac{\mu}{v}{u_x}^2\cdot\frac{\kappa}{v}\theta_t\,\mathrm{d}x\,\mathrm{d}\tau\right| \le \varepsilon X + C_\varepsilon\int_0^t\int_0^1(1+\theta)^{q-3}{u_x}^4\,\mathrm{d}x\,\mathrm{d}\tau,$$

whose right-hand side is estimated from above by

$$\begin{aligned}(2.20)\qquad &\varepsilon X + C_\varepsilon|{u_x}^2|^{(0)}\int_0^t\|u_x\|^2\,\mathrm{d}\tau\\ &\le \varepsilon X + C_\varepsilon\Big(1+Z^{3/4}\Big)\end{aligned}$$

for $2\le q\le 3$, by

$$\begin{aligned}(2.21)\qquad &\varepsilon X + C_\varepsilon|1+\theta^{q-3}|^{(0)}|{u_x}^2|^{(0)}\int_0^t\|u_x\|^2\,\mathrm{d}\tau\\ &\le \varepsilon X + C_\varepsilon\Big(1+Y^{\frac{q-3}{2q+6}}+Y^{\frac{q-3}{2q+6}}Z^{3/4}+Z^{3/4}\Big)\\ &\le \varepsilon(X+Y)+C_\varepsilon\big(1+Z^\delta\big)\end{aligned}$$

with a number δ ($0<\delta<1$) for $3<q<4$, and by

$$\begin{aligned}(2.22)\qquad &\varepsilon X + C_\varepsilon\big|\big(1+\theta^{q-3}\big)u_x\big|^{(0)}\int_0^t\|u_x\|^3_{L^3(\Omega)}\,\mathrm{d}\tau\\ &\le \varepsilon X + C_\varepsilon\Big(1+Y^{\frac{q-3}{2q+6}}+Y^{\frac{q-3}{2q+6}}Z^{3/8}+Z^{3/8}\Big)\\ &\le \varepsilon(X+Y)+C_\varepsilon\Big(1+Z^{3/4}\Big)\end{aligned}$$

for $q\ge4$ in virtue of (2.1). Combining (2.7)–(2.15), (2.18)–(2.22) and taking ε small, we obtain (2.6). □

Since the regularity of the solution is not sufficient, the following arguments are formal. However, one can justify them by using the method of difference quotients or mollifiers.

Lemma 4. *If $q \geq 3$ and $0 \leq \beta < q+9$, then for any $t \in [0,T]$*

$$\|u_x, u_{xx}, u_t, \theta_x, \theta_{xx}, \theta_t, z_x, z_{xx}, z_t\|^2 + \int_0^t \|u_{xt}, \theta_{xt}, z_{xt}\|^2 \, \mathrm{d}\tau \leq C_T; \tag{2.23}$$

$$|u_x|^{(0)},\ |u|^{(0)},\ |\theta|^{(0)} \leq C_T; \tag{2.24}$$

$$\theta(x,t) \geq C_T \quad \textit{for any } (x,t) \in \overline{Q_T}. \tag{2.25}$$

Proof. Differentiating $(1.1)^2$ with respect to t, multiplying it by u_t and integrating it with respect to x, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \frac{1}{2}{u_t}^2 \, \mathrm{d}x + \int_0^1 \frac{\mu}{v}{u_{xt}}^2 \, \mathrm{d}x = \int_0^1 \left(p_t u_{xt} + \frac{\mu}{v^2}{u_x}^2 u_{xt}\right) \mathrm{d}x.$$

Since $p_t = \left(\frac{R}{v} + \frac{4}{3}a\theta^3\right)\theta_t - \frac{R}{v^2}\theta u_x$, we get for $q \geq 3$

$$\begin{aligned} \|u_t\|^2 + \int_0^t \|u_{xt}\|^2 \, \mathrm{d}\tau &\leq C\left[1 + \int_0^t \int_0^1 \left({p_t}^2 + {u_x}^4\right) \mathrm{d}x \, \mathrm{d}\tau\right] \\ &\leq C\left[\int_0^t \int_0^1 \left(1+\theta^6\right){\theta_t}^2 \, \mathrm{d}x \, \mathrm{d}\tau + |{u_x}^2|^{(0)} \int_0^t (\|\theta\|^2 + \|u_x\|^2) \, \mathrm{d}\tau\right] \\ &\leq C\Big(1 + X + Z^{3/4}\Big) \leq C\big(1 + Z^\delta\big) \end{aligned} \tag{2.26}$$

from Lemma 3. By squaring $(1.1)^2$ and noting $p_x = \left(\frac{R}{v} + \frac{4}{3}a\theta^3\right)\theta_x - \frac{R}{v^2}\theta v_x$, it follows from (2.26) that for any $t \in [0,T]$

$$\begin{aligned} \|u_{xx}\|^2(t) &\leq C\left[1 + \|u_t\|^2 + \int_0^1 \left(1+\theta^6\right){\theta_x}^2 \, \mathrm{d}x + \Big(|\theta^2|^{(0)} + |{u_x}^2|^{(0)}\Big)\|v_x\|^2\right] \\ &\leq C\big(1 + Y + Z^\delta\big). \end{aligned}$$

This implies $Z \leq C\big(1 + Z^\delta\big)$, and hence, we conclude that Z is bounded. Then one can see from (2.5), (2.6) and (2.26) that $\|u_x, u_{xx}, u_t, \theta_x\|$, $|u, \theta|^{(0)}$ and $\int_0^t \|u_{xt}, \theta_t\|^2 \, \mathrm{d}\tau$ are also bounded. For the boundedness of other quantities in (2.23)–(2.25), see the proof of Lemmas 10–13 in [12]. □

The Hölder estimates are obtained from the classical Schauder estimates by using Lemmas 1–4 (see [12] for details).

References

[1] B. Ducomet, Some stability results for reactive Navier-Stokes-Poisson systems, in *Evolution equations: existence, regularity and singularities* (*Warsaw,* 1998), 83–118, Polish Acad. Sci., Polish Scientific Publishers, Warsaw, 2000.

[2] B. Ducomet and A. Zlotnik, Lyapunov functional method for 1D radiative and reactive viscous gas dynamics, Arch. Ration. Mech. Anal. **177** (2005), no. 2, 185–229.

[3] B. Ducomet and A. Zlotnik, On the large-time behavior of 1D radiative and reactive viscous flows for higher-order kinetics, Nonlinear Anal. **63** (2005), no. 8, 1011–1033.

[4] A. V. Kazhikhov, To the theory of boundary value problems for equations of a one-dimensional non-stationary motion of a viscous heat-conductive gas, Din. Sploshn. Sredy **50** (1981), 37–62. (in Russian).

[5] A. V. Kazhikhov and V. V. Shelukhin, Unique global solution with respect to time of the initial-boundary value problems for one-dimensional equations of a viscous gas, J. Appl. Math. Mech. **41** (1977), no. 2, 273–282.; translated from Prikl. Mat. Meh. **41** (1977), no. 2, 282–291.

[6] O. A. Ladyženskaja, V. A. Solonnikov and N. N. Ural'ceva, *Linear and quasi-linear equations of parabolic type*, Transl. Math. Monogr., vol. 23, Amer. Math. Soc., Providence, RI, 1968.

[7] T. Nagasawa, On the outer pressure problem of the one-dimensional polytropic ideal gas, Japan J. Appl. Math. **5** (1988), no. 1, 53–85.

[8] P. Secchi, On the motion of gaseous stars in the presence of radiation, Comm. Partial Differential Equations **15** (1990), no. 2, 185–204.

[9] S. F. Shandarin and Ya. B. Zel'dovich, The large-scale structure of the universe: turbulence, intermittency, structures in a self-gravitating medium, Rev. Modern Phys. **61** (1989), no. 2, 185–220.

[10] A. Tani, On the first initial-boundary value problem of compressible viscous fluid motion, Publ. Res. Inst. Math. Sci. **13** (1977), 193–253.

[11] A. Tani, On the free boundary value problem for compressible viscous fluid motion, J. Math. Kyoto Univ. **21** (1981), no. 4, 839–859.

[12] M. Umehara and A. Tani, Global solution to the one-dimensional equations for a self-gravitating viscous radiative and reactive gas, J. Differential Equations **234** (2007), no. 2, 439–463.

[13] M. Umehara and A. Tani, Temporally global solution to the equations for a spherically symmetric viscous radiative and reactive gas over the rigid core, Anal. Appl. **6** (2008), 183–211.