Free and non-free multiplicity on the deleted A_3 arrangement

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Abstract: We give the first complete classification of free and non-free multiplicities on an arrangement, called the deleted A_3 arrangement, which admits both of them.

Key words: Multiarrangement; free multiplicity; deleted A_3 arrangement.

0. Introduction. Let V be an ℓ-dimensional vector space over a field $\mathbf{K}$, $\{x_1, \ldots, x_\ell\}$ be a basis for the dual vector space V^* and $S := \mathrm{Sym}(V^*) \simeq \mathbf{K}[x_1, \ldots, x_\ell]$. $\mathrm{Der}_{\mathbf{K}}(S)$ denotes the S-module of $\mathbf{K}$-linear derivations of S, i.e., $\mathrm{Der}_{\mathbf{K}}(S) = \bigoplus_{i=1}^{\ell} S \cdot \partial_{x_i}$. We say a non-zero element $\theta = \sum_{i=1}^{\ell} f_i \partial_{x_i} \in \mathrm{Der}_{\mathbf{K}}(S)$ is *homogeneous of degree* p if f_i is zero or homogeneous of degree p for each i.

A *hyperplane arrangement* $\mathcal{A}$ (or simply an *arrangement*) is a finite collection of affine hyperplanes in V. If each hyperplane in $\mathcal{A}$ contains the origin, we say that $\mathcal{A}$ is *central*. In this article we assume that all arrangements are central. An *ℓ-arrangement* is the arrangement in the ℓ-dimensional vector space. A *multiplicity* m on an arrangement $\mathcal{A}$ is a map $m : \mathcal{A} \to \mathbf{Z}_{>0}$ and a pair $(\mathcal{A}, m)$ is called a *multiarrangement*. $|m|$ denotes the sum of the multiplicity $\sum_{H \in \mathcal{A}} m(H)$. When $m \equiv 1$, $(\mathcal{A}, m)$ is the same as a hyperplane arrangement $\mathcal{A}$ and sometimes called a *simple arrangement*. For each hyperplane $H \in \mathcal{A}$ fix a linear form $\alpha_H \in V^*$ such that $\ker(\alpha_H) = H$. Put $Q(\mathcal{A}, m) := \prod_{H \in \mathcal{A}} \alpha_H^{m(H)}$. The main object in this article is a *logarithmic derivation module* $D(\mathcal{A}, m)$ of $(\mathcal{A}, m)$ defined by

$$D(\mathcal{A}, m) := \{\theta \in \mathrm{Der}_{\mathbf{K}}(S) | \theta(\alpha_H) \in S \cdot \alpha_H^{m(H)} \text{ for all } H \in \mathcal{A}\}.$$

A multiarrangement $(\mathcal{A}, m)$ is called *free* if $D(\mathcal{A}, m)$ is a free S-module of rank ℓ. If $(\mathcal{A}, m)$ is free, then there exists a homogeneous free basis $\{\theta_1, \ldots, \theta_\ell\}$ for $D(\mathcal{A}, m)$. Then we define the *exponents* of a free multiarrangement $(\mathcal{A}, m)$ by $\exp(\mathcal{A}, m) := (\deg(\theta_1), \ldots, \deg(\theta_\ell))$. The exponents are independent of a choice of a basis.

Originally, a multiarrangement was defined by Ziegler in [8] and used effectively in the studies of hyperplane arrangements, e.g., in [5] and [6]. However, very few have been known about the freeness and non-freeness of multiarrangements. Recently, some theorems to consider the freeness of multiarrangements are developed in [1] and [2]. In these papers, a concept of free multiplicities is introduced. For a simple arrangement $\mathcal{A}$, we say a multiplicity m on $\mathcal{A}$ is *free* if the multiarrangement $(\mathcal{A}, m)$ is free. For example, every multiplicity is free on Boolean arrangements, and no multiplicity is free on a generic arrangement (see [2] and [7]). However, on an arrangement which admits both free and non-free multiplicity, only a partial classification of multiplicities is known. For example, Coxeter arrangements of type A_3 admits both free and non-free multiplicity, but such a classification is not known. Hence to consider the behavior, geometry and combinatorics of free and non-free multiplicities in $\mathbf{Z}_{>0}^{|\mathcal{A}|}$ is a new problem in the study of arrangements. In this article, we give the first complete classification of the freeness of all multiplicities on an arrangement which admits both free and non-free multiplicity. Let us fix $\ell = 3$ and a basis $\{x, y, z\}$ for V^*.

Definition 0.1. *An arrangement $\mathcal{A}$ is called the deleted A_3 arrangement if it is defined by*

$$Q(\mathcal{A}) = xy(x-y)(x-z)(y-z).$$

This is a free arrangement with $\exp(\mathcal{A}) = (1, 2, 2)$, but Ziegler proved in [8] that the constant multiplicity $m \equiv 2$ is not a free multiplicity. So the deleted A_3 arrangement admits both free and non-free multiplicity. Since this arrangement is close to the Coxeter arrangement of type A_3 and consists of only five planes, it is natural to consider the classification of multiplicities from the viewpoint of freeness. Our classification is as follows:

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Theorem 0.2. *Let $\mathcal{A}$ be the deleted A_3 arrangement and $m = [a, b, c, d, e]$ be a multiplicity on $\mathcal{A}$ defined by*

$$Q(\mathcal{A}, m) = (y - z)^a y^b (x - y)^c x^d (x - z)^e.$$

Then m is a free multiplicity if and only if $c \geq a + e - 1$ or $c \geq b + d - 1$.

Remark 0.3. *Theorem 0.2 implies, if we identify all the multiplicities on the deleted A_3 arrangement with $\mathbf{Z}^5_{>0} = \{(a, b, c, d, e)\}$ (i.e., the moduli space of multiplicities on $\mathcal{A}$), then the set of free multiplicities consists of three chambers of the complement of the arrangement in $\mathbf{Z}^5_{>0}$ defined by*

$$\left(c - a - e + \frac{3}{2}\right)\left(c - b - d + \frac{3}{2}\right) = 0.$$

Also note that a choice of such an arrangement is not unique.

Theorem 0.2 and other classifications of free multiplicities on Boolean arrangements or generic arrangements in [1], [2] and [7] pose the following question.

Question. For any arrangement $\mathcal{A}$, does the set of free multiplicities on $\mathcal{A}$ consist of chambers of some hyperplane arrangement in $\mathbf{Z}^{|\mathcal{A}|}_{>0}$?

The organization of this article is as follows. In Section 1 we introduce some results and notation which will be used in this article. In Section 2 we prove Theorem 0.2.

1. Preliminaries. In this section we fix the notation and introduce some results. To prove Theorem 0.2, we often use the following three results:

Theorem 1.1 [4]**.** *Let $\mathcal{A} = \{H_1, H_2, H_3\}$ be a 2-arrangement of three lines and k be a multiplicity on $\mathcal{A}$ with $k(H_i) = k_i$ $(i = 1, 2, 3)$. Assume that $k_3 \geq \max\{k_1, k_2\}$.*

(a) *If $k_3 \leq k_1 + k_2 + 1$, then it holds that*

$$|d_1 - d_2| = \begin{cases} 0 & \text{if } |k| \text{ is even}, \\ 1 & \text{if } |k| \text{ is odd}, \end{cases}$$

where $(d_1, d_2) = \exp(\mathcal{A}, k)$.

(b) *If $k_3 > k_1 + k_2 + 1$, then $\exp(\mathcal{A}, k) = (k_1 + k_2, k_3)$.*

Theorem 1.2 [1; Corollary 4.6]**.** *If a multiarrangement $(\mathcal{A}, m)$ is free, then $GMP(k) = LMP(k)$ $(1 \leq k \leq \ell)$, where $GMP(k)$ is the k-th global mixed product of $(\mathcal{A}, m)$ and $LMP(k)$ is the k-th local mixed product of $(\mathcal{A}, m)$.*

Theorem 1.3 [2; Theorem 5.10]**.** *Let $(\mathcal{A}, m)$ be a multiarrangement such that $\mathcal{A}$ is supersolvable with a filtration $\mathcal{A} = \mathcal{A}_r \supset \mathcal{A}_{r-1} \supset \cdots \supset \mathcal{A}_2 \supset \mathcal{A}_1$ and $r \geq 2$. Let m_i denote the multiplicity $m|_{\mathcal{A}_i}$ and $\exp(\mathcal{A}_2, m_2) = (d_1, d_2, 0, \ldots, 0)$. Assume that for each $H' \in \mathcal{A}_d \setminus \mathcal{A}_{d-1}$, $H'' \in \mathcal{A}_{d-1}$ $(d = 3, \ldots, r)$ and $X := H' \cap H''$, it holds that*

$$\mathcal{A}_X = \{H', H''\}$$

or that

$$m(H'') \geq \sum_{X \subset H \in (\mathcal{A}_d \setminus \mathcal{A}_{d-1})} m(H) - 1.$$

Then $(\mathcal{A}, m)$ is free with $\exp(\mathcal{A}, m) = (d_1, d_2, |m_3| - |m_2|, \ldots, |m_r| - |m_{r-1}|, 0, \ldots, 0)$.

For the details and notation of these theorems, see [1,2,4]. Note that the deleted A_3 arrangement is supersolvable. Theorem 1.2 is used to show the non-freeness of a multiarrangement. To apply it, we need some elementary results on number theory. From now on, let us assume $\ell = 3$ and fix a coordinate system $\{x, y, z\}$ for V^*. For the rest of this article we only consider the 2nd mixed products. Hence $LMP(\mathcal{A}, m)$ denotes the 2nd local mixed product of $(\mathcal{A}, m)$, and $GMP(\mathcal{A}, m)$ the 2nd global mixed product of $(\mathcal{A}, m)$ if it is free. In other words,

$$LMP(\mathcal{A}, m) = \sum_{X \in L(\mathcal{A})_2} d_1^X d_2^X,$$

where $L(\mathcal{A})_2$ consists of elements in the intersection lattice $L(\mathcal{A})$ of $\mathcal{A}$ (e.g., see [3]) such that $\operatorname{codim}_V(X) = 2$ and $\exp(\mathcal{A}_X, m|_{\mathcal{A}_X}) = (d_1^X, d_2^X, 0)$ for $X \in L(\mathcal{A})_2$. Moreover, if $(\mathcal{A}, m)$ is free with $\exp(\mathcal{A}, m) = (d_1, d_2, d_3)$, then

$$GMP(\mathcal{A}, m) = d_1 d_2 + d_2 d_3 + d_3 d_1.$$

Sometimes for the triple of integers (d_1, d_2, d_3), $GMP(d_1, d_2, d_3)$ denotes $d_1 d_2 + d_2 d_3 + d_3 d_1$. Let us agree that $(d_1, d_2, d_3)_{\leq}$ denotes the integers d_1, d_2, d_3 with $d_1 \leq d_2 \leq d_3$.

Lemma 1.4. *Let us put $m_0 := \max\{m(H) \mid H \in \mathcal{A}\}$ for a free multiarrangement $(\mathcal{A}, m)$ with $\exp(\mathcal{A}, m) = (d_1, d_2, d_3)_{\leq}$. Then $d_3 \geq m_0$.*

Proof. We may assume that $m_0 = m(\{x = 0\})$. If $d_3 < m_0$, then all elements in $D(\mathcal{A}, m)$ can be expressed as $f_y \partial_y + f_z \partial_z$ for $f_y, f_z \in S$, which contradicts to the freeness of $(\mathcal{A}, m)$. □

For a rational number $\alpha \in \mathbf{Q}$, let $\lceil \alpha \rceil$ denote the smallest integer which is larger than or equal to

α, and $\lfloor \alpha \rfloor$ the largest integer which is smaller than or equal to α. Then the proofs of the following two results are easy, so left to the reader.

Lemma 1.5. *Let $(\mathcal{A}, m)$ be a free multiarrangement with $\exp(\mathcal{A}, m) = (d_1, d_2, d_3)_{\le}$, $\mathcal{B} \subset \mathcal{A}$ be a subarrangement of $\mathcal{A}$ and m' be a multiplicity on $\mathcal{B}$ such that $m'(H) \le m(H)$ for all $H \in \mathcal{B}$. Assume that $(\mathcal{B}, m')$ is a free submultiarrangement of $(\mathcal{A}, m)$ with $\exp(\mathcal{B}, m') = (e_1, e_2, e_3)_{\le}$. Put $n := |m| - |m'|$ and assume that*

$$e_3 \ge \left\lceil \frac{|m|}{3} \right\rceil,\ e_2 \ge \left\lfloor \frac{|m| - e_3}{2} \right\rfloor.$$

Then $GMP(d_1, d_2, d_3) \le GMP(e_1 + n, e_2, e_3)$.

Lemma 1.6. *Let $(\mathcal{A}, m)$ be a free multiarrangement with $\exp(\mathcal{A}, m) = (d_1, d_2, d_3)_{\le}$. If $\max\{m(H) | H \in \mathcal{A}\} = a \ge \left\lceil \frac{|m|}{3} \right\rceil$, then*

$$GMP(d_1, d_2, d_3) \le GMP\left(a, \left\lceil \frac{|m| - a}{2} \right\rceil, \left\lfloor \frac{|m| - a}{2} \right\rfloor\right).$$

2. Proof of Theorem 0.2. In this section we prove Theorem 0.2. From now on, let $\mathcal{A}$ be the deleted A_3 arrangement and $m = [a, b, c, d, e]$ a multiplicity on $\mathcal{A}$ as in the statement of Theorem 0.2. For the proof, we introduce the following definition.

Definition 2.1.

(1) *Let $\exp[a, c, e]$ $(resp : \exp[b, c, d])$ denote the exponents of a 2-multiarrangement defined by $(y - z)^a (x - y)^c (x - z)^e = 0$ $(resp : y^b (x - y)^c x^d = 0)$. Note that these two are the submultiarrangements of $(\mathcal{A}, m)$. Also note that these are the same multiarrangement as in Theorem 1.1.*

(2) *If we put $\exp[a, c, e] = (d_1, d_2)$ and $\exp[b, c, d] = (e_1, e_2)$, then define $[a, c, e] := d_1 \times d_2$ and $[b, c, d] := e_1 \times e_2$. We say $[a, c, e]$ $(resp : [b, c, d])$ is balanced if the condition (a) in Theorem 1.1 is satisfied by the multiarrangement $(y - z)^a (x - y)^c (x - z)^e = 0$ $(resp : y^b (x - y)^c x^d = 0)$.*

Now let us prove Theorem 0.2. First we show the condition in Theorem 0.2 is a sufficient condition.

Proposition 2.2. *If $c \ge a + e - 1$ or $c \ge b + d - 1$, then $(\mathcal{A}, m)$ is free.*

Proof. Assume $c \ge a + e - 1$. Consider a supersolvable filtration $\mathcal{A}_3 \supset \mathcal{A}_2 \supset \mathcal{A}_1$ of $\mathcal{A}$ defined by

$$\mathcal{A}_1 := \{x = 0\},$$
$$\mathcal{A}_2 := \{xy(x - y) = 0\},$$
$$\mathcal{A}_3 := \{xy(x - y)(x - z)(y - z) = 0\}.$$

To complete the proof, apply Theorem 1.3. The same argument is valid when $c \ge b + d - 1$. In particular, $\exp(\mathcal{A}, m)$ can be seen by Theorem 1.1 and Theorem 1.3. $\square$

Hence, for the rest of this section, we assume the following condition:

$$c \le a + e - 2 \text{ and } c \le b + d - 2. \tag{2.1}$$

Note that, in particular, $c < \left\lfloor \frac{|m|}{3} \right\rfloor$. We show that $(\mathcal{A}, m)$ is not free under the condition (2.1) by using Theorem 1.2 and the related non-freeness criterion in [1]. After an appropriate change of coordinates, we may assume that

$$a \ge e,\ b \ge d, \text{ and } a \ge b.$$

Let us define a submultiarrangement $(\mathcal{B}, m')$ of $(\mathcal{A}, m)$ by

$$Q(\mathcal{B}, m') := (y - z)^a y^b (x - y)^c (x - z)^e.$$

Then $\mathcal{B}$ has a following supersolvable filtration:

$$\mathcal{B}_1 := \{y - z = 0\},$$
$$\mathcal{B}_2 := \{(y - z)(x - z)(x - y) = 0\},$$
$$\mathcal{B}_3 := \mathcal{B} = \{(y - z)(x - z)(x - y)y = 0\}.$$

Hence Theorem 1.3 implies that $(\mathcal{B}, m')$ is free with $\exp(\mathcal{B}, m') = (\exp[a, c, e], d)$.

Lemma 2.3. *Assume that $a \ge \left\lceil \frac{|m|}{3} \right\rceil$ and $b \ge \left\lfloor \frac{|m|}{3} \right\rfloor$. Then $(\mathcal{A}, m)$ is not free.*

Proof. Let us assume that $(\mathcal{A}, m)$ is free with $\exp(\mathcal{A}, m) = (d_1, d_2, d_3)$. Since $c + d + e \le \left\lceil \frac{|m|}{3} \right\rceil$, $LMP(\mathcal{A}, m) = (a + b)(c + d + e) + ab + de$. Also, Theorem 1.1 and the assumption imply that $\exp(\mathcal{B}, m') = (a, c + e, b)$. By the assumption, $(\mathcal{B}, m')$ satisfies the condition of Lemma 1.5. Therefore $GMP(d_1, d_2, d_3) \le GMP(c + d + e, b, a)$. Hence $LMP(\mathcal{A}, m) - GMP(d_1, d_2, d_3) \ge de > 0$, which contradicts to Theorem 1.2. $\square$

Lemma 2.4. *Assume that $a \ge \left\lceil \frac{|m|}{3} \right\rceil$ and $b < \left\lfloor \frac{|m|}{3} \right\rfloor$. Then $(\mathcal{A}, m)$ is not free.*

Proof. Assume that $(\mathcal{A}, m)$ is free with $\exp(\mathcal{A}, m) = (d_1, d_2, d_3)$. Put $LMP := LMP(\mathcal{A}, m)$,

$GMP := GMP(\mathcal{A}, m)$ and $LG := LMP - GMP$. Note that $\max\{a, b, c, d, e\} = a$ by the assumption and (2.1).

Case 1. When $b + d \le \left\lfloor \frac{|m|}{3} \right\rfloor$ and $[a, c, e]$ is balanced. In this case, $\exp(\mathcal{B}, m') = (\exp[a, c, e], b)$. Hence the assumption and Lemma 1.5 imply $GMP(d_1, d_2, d_3) \le GMP(b + d, \exp[a, c, e])$. So

$$\begin{aligned} LG &\ge (a + e)(b + d) + [a, c, e] + [b, c, d] \\ &\quad - \{(a + c + e)(b + d) + [a, c, e]\} \\ &= [b, c, d] - c(b + d). \end{aligned} \tag{2.2}$$

We show that (2.2) is positive, which contradicts to Theorem 1.2. If $[b, c, d]$ is balanced, then (2.2) is positive because of (2.1). If $[b, c, d]$ is not balanced, then the assumption and Theorem 1.1 imply that $[b, c, d] = b(c + d)$. Since $c < c + d \le b < b + d$, we can see that (2.2) is positive.

Case 2. When $b + d \le \left\lfloor \frac{|m|}{3} \right\rfloor$, $[a, c, e] = a(c + e)$ and $c + e \ge b + d$. In this case $\exp(\mathcal{B}, m') = a, c + e, b)$. So Lemma 1.5 implies that $GMP(d_1, d_2, d_3) \le GMP(b + d, c + e, a)$. Hence

$$\begin{aligned} LG &\ge a(b + c + d + e) + [b, c, d] + e(b + d) \\ &\quad - \{a(b + c + d + e) + (c + e)(b + d)\} \\ &= [b, c, d] - c(b + d). \end{aligned} \tag{2.3}$$

By the same argument as above, we can see that (2.3) is positive, which is a contradiction.

Hence it suffices to show the non-freeness under the following two conditions:

(1) $b + d \le \left\lfloor \frac{|m|}{3} \right\rfloor$, $[a, c, e] = a(c + e)$ and $b + d > c + e$.

(2) $b + d > \left\lfloor \frac{|m|}{3} \right\rfloor$.

Note that the condition $b + d > \left\lfloor \frac{|m|}{3} \right\rfloor$ implies $[a, c, e] = a(c + e)$ and $b + d > c + e$.

Case 3. When $[a, c, e] = a(c + e)$, $b + d > c + e$ and $[b, c, d]$ is balanced. Lemma 1.6 and $\max\{a, b, c, d, e\} = a$ imply $GMP(d_1, d_2, d_3) \le GMP\left(a, \left\lceil \frac{|m| - a}{2} \right\rceil, \left\lfloor \frac{|m| - a}{2} \right\rfloor\right)$. Hence

$$\begin{aligned} LG &\ge a(b + c + d + e) + [b, c, d] + e(b + d) \\ &\quad - \left\{a(b + c + d + e) + \left\lceil \frac{|m| - a}{2} \right\rceil \left\lfloor \frac{|m| - a}{2} \right\rfloor\right\} \\ &\ge e(b + d) + \frac{(b + c + d)^2}{4} - \frac{(b + c + d + e)^2}{4} - \frac{1}{4} \\ &= e(b + d) - \frac{e(b + c + d)}{2} - \frac{e^2}{4} - \frac{1}{4} \\ &= \frac{2e(b + d - c - e) + e^2 - 1}{4} > 0, \end{aligned}$$

which is a contradiction.

Case 4. When $[a, c, e] = a(c + e)$, $b + d > c + e$, $b \ge c + d + 2$ and $b < \frac{|m| - a}{2}$. Note that $b + d \ge \left\lceil \frac{|m| - a}{2} \right\rceil$ because $b + d > c + e$. Hence $b \ge c + d + 2$ implies

$$\begin{aligned} LMP &= a(b + c + d + e) + e(b + d) + b(c + d) \\ &\ge a(b + c + d + e) + e(b + d) \\ &\quad + \left\lceil \frac{|m| - a}{2} \right\rceil \left(b + c + d - \left\lceil \frac{|m| - a}{2} \right\rceil\right). \end{aligned}$$

Since $GMP \le GMP\left(a, \left\lceil \frac{|m| - a}{2} \right\rceil, \left\lfloor \frac{|m| - a}{2} \right\rfloor\right)$, we have

$$\begin{aligned} LG &\ge \left\lceil \frac{|m| - a}{2} \right\rceil \left(b + c + d - \left\lceil \frac{|m| - a}{2} \right\rceil - \left\lfloor \frac{|m| - a}{2} \right\rfloor\right) \\ &\quad + e(b + d) \\ &= \left\lceil \frac{|m| - a}{2} \right\rceil (b + c + d + e - (|m| - a)) \\ &\quad + e\left(b + d - \left\lceil \frac{|m| - a}{2} \right\rceil\right) \\ &= e\left(b + d - \left\lceil \frac{|m| - a}{2} \right\rceil\right). \end{aligned}$$

Since $b + d \ge \left\lceil \frac{|m| - a}{2} \right\rceil$, the last equation above is positive unless $b + d = \left\lceil \frac{|m| - a}{2} \right\rceil$ and $c + e = \left\lfloor \frac{|m| - a}{2} \right\rfloor$. In this case, the same argument as Case 2 implies $LMP > GMP$.

Case 5. When $[a, c, e] = a(c + e)$, $b + d > c + e$, $b \ge c + d + 2$ and $b \ge \frac{|m| - a}{2}$. Lemma 1.5 implies $GMP \le GMP(c + d + e, b, a)$. Hence

$$LG \ge de > 0,$$

and the proof is completed. □

By Lemma 2.3 and 2.4, we may assume the following condition:

$$a, b, c, d, e < \left\lceil \frac{|m|}{3} \right\rceil. \tag{2.4}$$

Lemma 2.5. *Assume that* $b + d \leq \left\lceil \frac{|m|}{3} \right\rceil$ $\left(\textit{resp: } a + e \leq \left\lceil \frac{|m|}{3} \right\rceil\right)$*. Then* $(\mathcal{A}, m)$ *is not free.*

Proof. Let us put $LG = LMP(\mathcal{A}, m) - GMP(\mathcal{A}, m)$ again. Assume that $(\mathcal{A}, m)$ is free with $\exp(\mathcal{A}, m) = (d_1, d_2, d_3)$. If $b + d \leq \left\lceil \frac{|m|}{3} \right\rceil$, then $a + c + e \geq \left\lfloor \frac{2}{3}|m| \right\rfloor$. Also, the condition (2.4) implies $[a,c,e]$ is balanced. So $GMP = GMP(d_1, d_2, d_3) \leq GMP(b + d, \exp[a, c, e])$ by Lemma 1.5. Thus

$$\begin{aligned} LG &\geq (a + e)(b + d) + [a, c, e] + [b, c, d] \\ &\quad - \{[a, c, e] + (a + c + e)(b + d)\} \\ &= [b, c, d] - c(b + d). \end{aligned} \tag{2.5}$$

By the same arguments as in the proof of Lemma 2.4, we can see that (2.5) is positive, which contradicts to Theorem 1.2. □

Therefore we may assume that

$$\left\lceil \frac{|m|}{3} \right\rceil < a + e < \left\lfloor \frac{2}{3}|m| \right\rfloor - 1, \tag{2.6}$$

$$\left\lceil \frac{|m|}{3} \right\rceil < b + d < \left\lfloor \frac{2}{3}|m| \right\rfloor - 1. \tag{2.7}$$

Hence the next lemma completes the proof of Theorem 0.2.

Lemma 2.6. *Under the conditions* (2.1)*,* (2.4)*,* (2.6) *and* (2.7)*,* $(\mathcal{A}, m)$ *is not free.*

Proof. Assume that $(\mathcal{A}, m)$ is free with $\exp(\mathcal{A}, m) = (d_1, d_2, d_3)$. Note that $LMP := LMP(\mathcal{A}, m) = (a + e)(b + d) + [a, c, e] + [b, c, d]$. Put $GMP := GMP(\mathcal{A}, m)$.

Case 1. When $|m| = 3k$ $(k \in \mathbf{Z})$. By the assumptions, $a < k$, $b < k$, $a + e > k$, $b + d > k$. Hence, if we define a new multiplicity $\overline{m}$ by $Q(\mathcal{A}, \overline{m}) = (x - y)^c (y - z)^k y^k x^{b+d-k} (x - z)^{a+e-k}$, then $LMP \geq LMP(\mathcal{A}, \overline{m}) = 3k^2 + (a + e - k)(b + d - k)$. So $GMP \leq GMP(k, k, k) = 3k^2 < LMP$, which is a contradiction.

Case 2. When $|m| = 3k + 1$ $(k \in \mathbf{Z})$. By the assumptions, $a < k + 1$, $b < k + 1$, $a + e > k + 1$, $b + d > k + 1$. Hence, if we define a new multiplicity $\overline{m}$ by $Q(\mathcal{A}, \overline{m}) = (x - y)^c (y - z)^{k+1} y^k x^{b+d-k} (x - z)^{a+e-k-1}$, then $LMP \geq LMP(\mathcal{A}, \overline{m}) = 3k^2 + 2k + (a + e - k - 1)(b + d - k)$. So $GMP \leq GMP(k + 1, k, k) = 3k^2 + 2k < LMP$, which is a contradiction.

Case 3. When $|m| = 3k + 2$ $(k \in \mathbf{Z})$. By the assumptions, $a < k + 1$, $b < k + 1$, $a + e > k + 1$, $b + d > k + 1$. Hence, if we define a new multiplicity $\overline{m}$ by $Q(\mathcal{A}, \overline{m}) = (x - y)^c (y - z)^{k+1} y^{k+1} x^{b+d-k-1} (x - z)^{a+e-k-1}$, then $LMP \geq LMP(\mathcal{A}, \overline{m}) = 3k^2 + 4k + 1 + (a + e - k - 1)(b + d - k - 1)$. So $GMP \leq GMP(k + 1, k, k) = 3k^2 + 4k + 1 < LMP$, which is a contradiction. □

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