Estimates for convergence rate of an n-Ginzburg-Landau type minimizer

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Abstract: The paper is concerned with the asymptotic analysis of a minimizer of an n-Ginzburg-Landau type functional. The convergence rate of the module of minimizers is presented when the parameter ε goes to zero. This conclusion shows that the functional converges to $\frac{1}{n}\int|\nabla u_n|^n$ locally when $\varepsilon \to 0$, where u_n is an n-harmonic map.

Key words: n-Ginzburg-Landau type functional; asymptotic analysis; regularized minimizer; convergence rate; n-harmonic map.

1. Introduction. Let $G \subset \mathbf{R}^n$ $(n \geq 3)$ be a bounded and simply connected domain with smooth boundary ∂G. g is a smooth map from ∂G into S^{n-1} and satisfies $\deg(g, \partial G) = d \neq 0$. Without loss of generality, we may assume $d > 0$. We are concerned with the asymptotic behavior of minimizers of the n-Ginzburg-Landau type functional

$$E_\varepsilon(u, G) = \frac{1}{n}\int_G |\nabla u|^n + \frac{1}{4\varepsilon^n}\int_G (1-|u|^2)^2,$$

in the function class $W = \{v \in W^{1,n}(G, \mathbf{R}^n); v|_{\partial G} = g\}$ when $\varepsilon \to 0^+$. In the case of $n = 2$, the asymptotic behavior of minimizers in W has been studied in many papers such as [1, 6]. It turns out to be that, there exist d points $\{a_i\}_{i=1}^d$ in G, such that for any compact subset K of $G \setminus \{a_i\}_{i=1}^d$, there holds a convergence

$$(1.1) \quad \lim_{\varepsilon \to 0} \frac{1-|u_\varepsilon|^2}{\varepsilon^2} = |\nabla u_2|^2, \quad in \quad C^k(K)$$

for any $k \geq 1$, where u_2 is a harmonic map on $G \setminus \{a_i\}_{i=1}^d$ (cf. [1, Theorem VI.1,(11)]).

When $n \geq 3$, the convergence of the minimizer u_ε of $E_\varepsilon(u, G)$ in W is a problem introduced in [1]. M.C.Hong studied this problem partly (cf. [3]). He proved that as $\varepsilon \to 0$, there exist a subsequence u_{ε_k} of the *regularized minimizer* u_ε and $\{a_1, a_2, ..., a_J\} \subset \overline{G}$, $J \in \mathbf{N}$, such that $u_{\varepsilon_k} \to u_n$ weakly in $W^{1,n}_{loc}(G \setminus \{a_1, a_2, ..., a_J\}, \mathbf{R}^n)$, where u_n is an n-harmonic map on $G \setminus \{a_1, a_2, ..., a_J\}$. Furthermore, [2] shows that $J = d$, $\deg(u_n, a_j) = 1$ with all $j = 1, 2, \cdots, d$, and when $\varepsilon \to 0$,

$$(1.2) \quad u_{\varepsilon_k} \to u_n, \quad in \quad W^{1,n}_{loc}(G \setminus \cup_{i=1}^d \{a_i\}, \mathbf{R}^n).$$

Other related work can be seen in [5, 7].

There may be several minimizers of $E_\varepsilon(u, G)$ in W, one of which, named the *regularized minimizer*, is the limit of the minimizer u_ε^τ of the following regularized functional in W

$$E_\varepsilon^\tau(u, G) = \frac{1}{n}\int_G (|\nabla u|^2 + \tau)^{n/2} + \frac{1}{4\varepsilon^n}\int_G (1-|u|^2)^2$$

in the $W^{1,n}$ sense when $\tau \to 0^+$. Moreover, (5.4) in [4] shows that there exists a subsequence of u_ε^τ, which is still denoted by itself, such that

$$(1.3) \quad \lim_{\tau \to 0} u_\varepsilon^\tau = u_\varepsilon, \quad \text{in } C^{1,\alpha}_{loc}(G \setminus \cup_{i=1}^d \{a_i\}, \mathbf{R}^n),$$

where $\alpha \in (0,1)$. From [3, Theorem 2.2], we can also deduce $|u_\varepsilon| \leq 1$ on $\overline{G}$.

In this paper, we will estimate the convergence rate of $|u_\varepsilon|$ to 1 on an arbitrary compact subset K of $G \setminus \{a_j\}_{j=1}^d$ when $\varepsilon \to 0$.

Theorem 1.1. *Assume u_ε is a regularized minimizer of $E_\varepsilon(u, G)$ in W. Then for any compact subset K of $G \setminus (\cup_{j=1}^d \{a_j\})$, there exists a positive constant C, such that as $\varepsilon \in (0, \varepsilon_0)$,*

$$(1.4) \quad \int_K |\nabla |u_\varepsilon||^n + \frac{1}{\varepsilon^n}\int_K (1-|u_\varepsilon|^2)^2 \leq C\varepsilon^{\frac{2n}{n^2-2}},$$

$$(1.5) \quad \left|\int_K \left(\frac{1-|u_\varepsilon|^2}{\varepsilon^n} - |\nabla u_\varepsilon|^n\right) dx\right| \leq C\varepsilon^{\frac{2}{n^2-2}}$$

where ε_0 is sufficiently small. Furthermore, when $\varepsilon \to 0$,

$$(1.6) \quad E_\varepsilon(u_\varepsilon, K) \to \frac{1}{n}\int_K |\nabla u_n|^n,$$

where u_n is the n-harmonic map in (1.2).

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Remark. (i) From (5.1) in [4] and (1.3), we can also deduce that $\|1-|u_\varepsilon|^2\|_{L^\infty(K)} \le C\varepsilon^n$. This is the convergence rate of $1-|u_\varepsilon|$ to zero in the L^∞ sense. Estimation (1.4) implies the convergence rate in the $W^{1,n}$ sense.

(ii) Estimation (1.5), together with (1.2), implies the following conclusion as (1.1),

$$\lim_{\varepsilon\to 0}\frac{1-|u_\varepsilon|^2}{\varepsilon^n} = |\nabla u_n|^n, \quad in \quad L^1(K).$$

(iii) If we notice that

$$E_\varepsilon(u,K) = \frac{1}{n}\int_K (|\nabla|u||^2 + |u|^2|\nabla \tfrac{u}{|u|}|^2)^{n/2} + \frac{1}{4\varepsilon^n}\int_K (1-|u|^2)^2,$$

the estimation (1.4) and the convergence (1.6) show that the energy functional $E_\varepsilon(u_\varepsilon, K)$ concentrates to the term $\frac{1}{n}\int_K |\nabla \frac{u_\varepsilon}{|u_\varepsilon|}|^n$ when ε is sufficiently small.

2. Preliminaries.

Proposition 2.1. *Assume u_ε is a regularized minimizer of $E_\varepsilon(u,G)$ in W. Then for any compact subset K of $G\setminus(\cup_{j=1}^d\{a_j\})$, there exists a positive constant C, which is independent of $\varepsilon\in(0,1)$, such that*

$$E_\varepsilon(u_\varepsilon, K) \le C\varepsilon^{2/n} + \frac{1}{n}\int_K |\nabla \frac{u_\varepsilon}{|u_\varepsilon|}|^n. \tag{2.1}$$

Proof. Choose $R>0$ sufficiently small such that $B(x,3R)\subset G\setminus(\cup_{j=1}^d\{a_j\})$. By Lemma 3.7 in [2] we know that

$$|u_\varepsilon| \ge 1/2, \quad on \quad B(x,3R) \tag{2.2}$$

as ε is sufficiently small. Thus, we can write $w = \frac{u_\varepsilon}{|u_\varepsilon|}$ on $B(x,3R)$. On the other hand, by Proposition 3.8 in [2], there exists a constant $C>0$ (independent of ε) such that

$$E_\varepsilon(u_\varepsilon, B(x,3R)) \le C. \tag{2.3}$$

By (2.3) and the integral mean value theorem, there is a constant $r\in(2R,3R)$ such that

$$\frac{1}{n}\int_{\partial B(x,r)} |\nabla u_\varepsilon|^n + \frac{1}{4\varepsilon^n}\int_{\partial B(x,r)} (1-|u_\varepsilon|^2)^2 = C(R)E_\varepsilon(u_\varepsilon, B_{3R}\setminus B_{2R}) \le C. \tag{2.4}$$

Consider the functional

$$H(\rho,B) = \frac{1}{n}\int_B (|\nabla\rho|^2+1)^{n/2} + \frac{1}{2\varepsilon^n}\int_B (1-\rho)^2,$$

where $B = B(x,r)$. Clearly, the minimizer ρ_1 of $H(\rho,B)$ in $W^{1,n}_{|u_\varepsilon|}(B, \mathbf{R}^+\cup\{0\})$ exists and solves

$$-div(v^{(n-2)/2}\nabla\rho) = \frac{1}{\varepsilon^n}(1-\rho) \quad on \quad B, \tag{2.5}$$

$$\rho|_{\partial B} = |u_\varepsilon|, \tag{2.6}$$

where $v = |\nabla\rho|^2+1$. Since $1/2 < |u_\varepsilon| \le 1$, it follows from the maximum principle that on $\overline{B}$,

$$\frac{1}{2} < \rho_1 \le 1. \tag{2.7}$$

Applying (2.3) we see easily that

$$\begin{aligned} H(\rho_1,B) &\le H(|u_\varepsilon|,B) \\ &\le C(E_\varepsilon(u_\varepsilon,B)+1) \le C. \end{aligned} \tag{2.8}$$

Multiplying (2.5) by $(\nu\cdot\nabla\rho)$, where $\rho=\rho_1$, and integrating over B, we have

$$-\int_{\partial B} v^{(n-2)/2}(\nu\cdot\nabla\rho)^2 + \int_B v^{(n-2)/2}\nabla\rho\cdot\nabla(\nu\cdot\nabla\rho) = \frac{1}{\varepsilon^n}\int_B (1-\rho)(\nu\cdot\nabla\rho), \tag{2.9}$$

where ν denotes the unit outward normal vector on ∂B. Using (2.8) we obtain

$$\left|\int_B v^{(n-2)/2}\nabla\rho\cdot\nabla(\nu\cdot\nabla\rho)\right| \le C + \frac{1}{n}\int_{\partial B} v^{n/2}. \tag{2.10}$$

Combining (2.6), (2.4) and (2.8) we also have

$$\begin{aligned} &\left|\frac{1}{\varepsilon^n}\int_B (1-\rho)(\nu\cdot\nabla\rho)\right| \\ &\le \frac{1}{2\varepsilon^n}\int_B (1-\rho)^2|div\nu| + \frac{1}{2\varepsilon^n}\int_{\partial B}(1-\rho)^2 \\ &\le C. \end{aligned}$$

Substituting this and (2.10) into (2.9) yields

$$\left|\int_{\partial B} v^{(n-2)/2}(\nu\cdot\nabla\rho)^2\right| \le C + \frac{1}{n}\int_{\partial B} v^{n/2}. \tag{2.11}$$

Applying (2.6), (2.4) and (2.11), we obtain that for any $\delta\in(0,1)$,

$$\begin{aligned} &\int_{\partial B} v^{n/2} \\ &= \int_{\partial B} v^{(n-2)/2}[1+(\tau\cdot\nabla\rho)^2+(\nu\cdot\nabla\rho)^2] \\ &\le \int_{\partial B} v^{(n-2)/2} + \int_{\partial B} v^{(n-2)/2}(\nu\cdot\nabla\rho)^2 \\ &+ \left(\int_{\partial B} v^{n/2}\right)^{(n-2)/n}\left(\int_{\partial B}(\tau\cdot\nabla|u_\varepsilon|)^n\right)^{2/n} \\ &\le C(\delta) + \left(\frac{1}{n}+2\delta\right)\int_{\partial B} v^{n/2}, \end{aligned}$$

where τ denotes the unit tangent vector on ∂B. Hence, it follows, if we choose $\delta > 0$ sufficiently small, that

$$\int_{\partial B} v^{n/2} \le C. \tag{2.12}$$

Now we multiply both sides of (2.5) by $(1-\rho)$ and integrate over B. Then

$$\int_B v^{(n-2)/2}|\nabla\rho|^2 + \frac{1}{\varepsilon^n}\int_B (1-\rho)^2 = -\int_{\partial B} v^{(n-2)/2}(\nu\cdot\nabla\rho)(1-\rho).$$

Using this result, Hölder's inequality and (2.4), (2.6), (2.7), (2.12), we obtain

$$\begin{aligned}&\int_B v^{(n-2)/2}|\nabla\rho|^2 + \frac{1}{\varepsilon^n}\int_B (1-\rho)^2\\ \le{}& C\left|\int_{\partial B} v^{n/2}\right|^{(n-1)/n}\left|\int_{\partial B}(1-\rho)^2\right|^{1/n}\\ \le{}& C\left|\int_{\partial B}(1-|u_\varepsilon|)^2\right|^{1/n} \le C\varepsilon.\end{aligned} \tag{2.13}$$

Since u_ε is a minimizer of $E_\varepsilon(u,G)$ in W, we have $E_\varepsilon(u_\varepsilon,G)\le E_\varepsilon(U,G)$, where

$$U=\rho_1 ,w \quad on \quad B, \quad \left(w=\frac{u_\varepsilon}{|u_\varepsilon|}\right);$$

$$U = u_\varepsilon \quad on \quad G\setminus B.$$

Hence

$$\begin{aligned}E_\varepsilon(u_\varepsilon,B) &\le E_\varepsilon(\rho_1 w,B)\\ &= \frac{1}{n}\int_B(|\nabla\rho_1|^2+\rho_1^2|\nabla w|^2)^{n/2}\\ &\quad+\frac{1}{4\varepsilon^n}\int_B(1-\rho_1^2)^2.\end{aligned} \tag{2.14}$$

From the mean value theorem, it is deduced that

$$\begin{aligned}&\int_B(|\nabla\rho_1|^2+\rho_1^2|\nabla w|^2)^{n/2}dx\\ &-\int_B(\rho_1^2|\nabla w|^2)^{n/2}dx\\ ={}&\frac{n}{2}\int_B\int_0^1[((|\nabla\rho_1|^2+\rho_1^2|\nabla w|^2)s\\ &+\rho_1^2|\nabla w|^2(1-s))^{(n-2)/2}]ds|\nabla\rho_1|^2dx\\ \le{}& C\int_B(|\nabla\rho_1|^n+|\nabla\rho_1|^2|\nabla w|^{n-2})dx.\end{aligned} \tag{2.15}$$

According to Theorem 1.1 in [4], there exists a constant $C=C(R)>0$, which is independent of ε, such that

$$\sup_{B_{3R}}|\nabla u_\varepsilon| \le C(R). \tag{2.16}$$

Using (2.2) and (2.16), from (2.13) we can deduce that

$$\begin{aligned}&\int_B(|\nabla\rho_1|^n+|\nabla\rho_1|^2|\nabla w|^{n-2})\\ \le{}&\int_B(|\nabla\rho_1|^n+4^{n-2}|\nabla\rho_1|^2|u_\varepsilon|^{n-2}|\nabla w|^{n-2})\\ \le{}& C\int_B(|\nabla\rho_1|^n+|\nabla\rho_1|^2)\le C(\varepsilon+\varepsilon^{2/n}).\end{aligned}$$

Combining this with (2.14), (2.15), and using (2.13), we can derive

$$E_\varepsilon(u_\varepsilon,B)\le\frac{1}{n}\int_B\rho_1^n|\nabla w|^n + C\varepsilon^{2/n}.$$

Noting (2.7), we can see (2.1) by an argument of the finite covering.

3. Proof of Theorem 1.1.

Proof of (1.4). Assume u_ε is a regularized minimizer, and $B=B(x,r)$ is the ball introduced in §2. By Jensen's inequality, we have

$$\begin{aligned}E_\varepsilon(u_\varepsilon,B)\ge{}&\frac{1}{n}\int_B|\nabla h|^n+\frac{1}{n}\int_B h^n|\nabla w|^n\\ &+\frac{1}{4\varepsilon^n}\int_B(1-h^2)^2,\end{aligned}$$

where $h=|u_\varepsilon|$ and $w=\frac{u_\varepsilon}{|u_\varepsilon|}$. Thus, from (2.1) it follows that,

$$\begin{aligned}&\frac{1}{n}\int_B|\nabla h|^n+\frac{1}{n}\int_B(h^n-1)|\nabla w|^n\\ &+\frac{1}{4\varepsilon^n}\int_B(1-h^2)^2\\ &\le E_\varepsilon(u_\varepsilon,B)-\frac{1}{n}\int_B|\nabla w|^n\le C\varepsilon^{2/n}.\end{aligned} \tag{3.1}$$

Using (2.2) and (2.16), we have

$$\begin{aligned}&\frac{1}{n}\int_B(1-h^n)|\nabla w|^n\\ &\le\frac{4^n}{n}\int_B(1-h^n)|\nabla u_\varepsilon|^n\\ &\le C(R)\varepsilon^{n/2}(\frac{1}{\varepsilon^n}\int_B(1-h^2)^2)^{1/2}.\end{aligned} \tag{3.2}$$

From (2.3) it follows

$$\frac{1}{n}\int_B(1-h^n)|\nabla w|^n\le C\varepsilon^{n/2}. \tag{3.3}$$

Applying Young's inequality to (3.2), we also see that for any $\delta \in (0,1)$,

$$\begin{aligned}&\frac{1}{n}\int_B (1-h^n)|\nabla w|^n \\ &\le \delta\left(\frac{1}{\varepsilon^n}\int_B (1-h^2)^2\right) + C(\delta)\varepsilon^n.\end{aligned} \tag{3.4}$$

Substituting this into (3.1), we get

$$\int_B |\nabla h|^n + \frac{1}{\varepsilon^n}\int_B (1-h^2)^2 \le C(\varepsilon^n + \varepsilon^{2/n}).$$

Based on this result, we can prove (1.4) by induction. Assume for some $j \ge 1, r_j \in (2R, r)$ such that

$$\frac{1}{\varepsilon^n}\int_{B(x,r_j)} (1-h^2)^2 \le C\varepsilon^n + C\varepsilon^{\frac{2}{n}\sum_{i=1}^{j}(\frac{2}{n^2})^{i-1}}$$

holds. Thus, by the integral mean value theorem, there exists a constant $r_{j+1} \in (2R, r_j)$ such that

$$\begin{aligned}&\frac{1}{\varepsilon^n}\int_{\partial B(x,r_{j+1})} (1-h^2)^2 \\ &\le C(R)\varepsilon^{\frac{2}{n}\sum_{i=1}^{j}(\frac{2}{n^2})^{i-1}}.\end{aligned} \tag{3.5}$$

Denote $B_{j+1} = B(x, r_{j+1})$ and consider the functional

$$H(\rho, B_{j+1}) = \frac{1}{n}\int_{B_{j+1}}\left[(|\nabla\rho|^2+1)^{n/2} + \frac{1}{2\varepsilon^n}(1-\rho)^2\right].$$

Of course, the functional $H(\rho, B_{j+1})$ achieves its minimum in $W^{1,n}_{|u_\varepsilon|}(B_{j+1}, \mathbf{R}^+ \cup \{0\})$ at a function ρ_{j+1}. By the same derivation of (2.13), we can also deduce from (3.5) that

$$\begin{aligned}&\int_{B_{j+1}} v^{(n-2)/2}|\nabla\rho_{j+1}|^2 + \frac{1}{\varepsilon^n}\int_{B_{j+1}}(1-\rho_{j+1})^2 \\ &\le C\left|\int_{\partial B_{j+1}}(1-|u_\varepsilon|)^2\right|^{1/n} \le C\varepsilon^{\sum_{i=1}^{j+1}(\frac{2}{n^2})^{i-1}}.\end{aligned}$$

Thus, the result (2.1) can be improved as

$$E_\varepsilon(u_\varepsilon, B_{j+1}) \le C\varepsilon^{\frac{2}{n}\sum_{i=1}^{j+1}(\frac{2}{n^2})^{i-1}} + \frac{1}{n}\int_{B_{j+1}}|\nabla w|^n.$$

Applying Jensen's inequality and (3.4), we may rewrite (3.1) as

$$\begin{aligned}&\frac{1}{n}\int_{B_{j+1}}|\nabla h|^n + \frac{1}{4\varepsilon^n}\int_{B_{j+1}}(1-h^2)^2 \\ &\le C\varepsilon^{\frac{2}{n}\sum_{i=1}^{j+1}(\frac{2}{n^2})^{i-1}} + \frac{1}{n}\int_{B_{j+1}}(1-h^n)|\nabla w|^n \\ &\le C\varepsilon^{\frac{2}{n}\sum_{i=1}^{j+1}(\frac{2}{n^2})^{i-1}} + C(\delta)\varepsilon^n \\ &\quad +\delta\left(\frac{1}{\varepsilon^n}\int_{B_{j+1}}(1-h^2)^2\right)\end{aligned}$$

for any $\delta \in (0,1)$. If we choose δ small enough, then

$$\begin{aligned}&\int_{B_{j+1}}|\nabla h|^n + \frac{1}{\varepsilon^n}\int_{B_{j+1}}(1-h^2)^2 \\ &\le C\varepsilon^{\frac{2}{n}\sum_{i=1}^{j+1}(\frac{2}{n^2})^{i-1}} + C\varepsilon^n.\end{aligned} \tag{3.6}$$

In view of this, we can see that (3.6) always holds for any $j \ge 1$. Letting $j \to \infty$, we have

$$\begin{aligned}&\int_{B_{2R}}|\nabla h|^n + \frac{1}{\varepsilon^n}\int_{B_{2R}}(1-h^2)^2 \\ &\le C\varepsilon^{\frac{2}{n}\sum_{i=1}^{\infty}(\frac{2}{n^2})^{i-1}} + C\varepsilon^n \le C\varepsilon^{\frac{2n}{n^2-2}}.\end{aligned}$$

Thus (1.4) can be proved easily.

Proof of (1.5). Obviously, the minimizer u_ε^τ of the regularized functional $E_\varepsilon^\tau(u, G)$ in W satisfies the Euler-Lagrange equation

$$-div(v^{(n-2)/2}\nabla u) = \frac{1}{\varepsilon^n}u(1-|u|^2), \quad in \quad G,$$

where $v = |\nabla u|^2 + \tau$. Taking the inner product of both sides of the system above with u, we have

$$-div(v^{(n-2)/2}\nabla u)u = \frac{1}{\varepsilon^n}|u|^2(1-|u|^2),$$

where $u = u_\varepsilon^\tau$. Combining this with $\nabla(|u|^2) = 2u \cdot \nabla u$, and

$$\begin{aligned}&-div(v^{(n-2)/2}\nabla u)u \\ &= -div(v^{(n-2)/2}u\cdot\nabla u) + v^{(n-2)/2}|\nabla u|^2\end{aligned}$$

we obtain

$$\begin{aligned}&\frac{1}{\varepsilon^n}|u|^2(1-|u|^2) \\ &= v^{(n-2)/2}|\nabla u|^2 - \frac{1}{2}div(v^{(n-2)/2}\nabla(|u|^2)).\end{aligned}$$

Adding $\frac{1}{\varepsilon^n}(1-|u|^2)^2$ to both sides of the equality above, we get

$$\begin{aligned}&\frac{1}{\varepsilon^n}(1-|u|^2) - v^{(n-2)/2}|\nabla u|^2 \\ &= \frac{1}{\varepsilon^n}(1-|u|^2)^2 \\ &\quad -\frac{1}{2}div(v^{(n-2)/2}\nabla(|u|^2)).\end{aligned} \tag{3.7}$$

Similar to the derivations of (2.4), from (1.4), we can deduce that

$$\int_{\partial B}|\nabla|u_\varepsilon^\tau||^n \le C\varepsilon^{\frac{2n}{n^2-2}}, \tag{3.8}$$

where B is some ball in $B(x,3R)\setminus B(x,2R)$. Integrating (3.7) over B, we have

$$\left|\int_B\left[\frac{1}{\varepsilon^n}(1-|u|^2)-v^{(n-2)/2}|\nabla u|^2\right]\right| \leq \frac{1}{\varepsilon^n}\int_B(1-|u|^2)^2+\frac{1}{2}\left|\int_{\partial B}v^{(n-2)/2}\nabla(|u|^2)d\zeta\right|.$$

Letting $\tau\to 0$, and using (1.3) we can see that

$$\left|\int_B\left[\frac{1}{\varepsilon^n}(1-|u_\varepsilon|^2)-|\nabla u_\varepsilon|^n\right]\right| \leq \frac{1}{\varepsilon^n}\int_B(1-|u_\varepsilon|^2)^2 +\frac{1}{2}\left|\int_{\partial B}|\nabla u_\varepsilon|^{n-2}\nabla(|u_\varepsilon|^2)d\zeta\right|.$$

By applying (1.4), Hölder's inequality and (2.16), (3.8), we get

$$\left|\int_B\left[\frac{1}{\varepsilon^n}(1-|u_\varepsilon|^2)-|\nabla u_\varepsilon|^n\right]\right|\leq C\varepsilon^{\frac{2}{n^2-2}}.$$

Thus (1.5) is deduced by an argument of the finite covering.

Proof of (1.6). At first, (3.1) implies

$$\begin{aligned}0 &\leq E_\varepsilon(u_\varepsilon,B)-\frac{1}{n}\int_B|\nabla w|^n +\frac{1}{n}\int_B(1-h^n)|\nabla w|^n\\ &\leq C\varepsilon^{2/n}+\frac{1}{n}\int_B(1-h^n)|\nabla w|^n.\end{aligned}$$

Combining this with (3.3), and letting $\varepsilon\to 0$, we have

$$E_\varepsilon(u_\varepsilon,B)-\frac{1}{n}\int_B|\nabla w|^n\to 0. \tag{3.9}$$

Next, we observe that

$$\begin{aligned}&\left|\int_B(|\nabla u_\varepsilon|^n-|\nabla w|^n)dx\right|\\ &\leq\left|\int_B(|\nabla u_\varepsilon|^n-h^n|\nabla w|^n)\right| +\left|\int_B|\nabla w|^n(1-h^n)\right|\\ &=I_1+I_2.\end{aligned}\tag{3.10}$$

In view of (3.3), we have $\lim_{\varepsilon\to 0}I_2=0$. In addition, the mean value theorem implies

$$\begin{aligned}I_1 &\leq C\int_B\left(\int_0^1[s|\nabla h|^2 + (1-s)h^2|\nabla w|^2]^{(n-2)/2}ds\right)|\nabla h|^2dx\\ &\leq C\left(\int_B|\nabla u_\varepsilon|^n\right)^{(n-2)/n}\left(\int_B|\nabla h|^n dx\right)^{2/n}.\end{aligned}$$

This result, together with (2.3) and (1.4), implies $\lim_{\varepsilon\to 0}I_1=0$. Substituting these into (3.10), and using (1.2) we deduce that

$$\lim_{\varepsilon\to 0}\int_B|\nabla w|^n=\int_B|\nabla u_n|^n.$$

Combining this with (3.9) yields (1.6).

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