Exact WKB analysis for the degenerate third Painlevé equation of type (D_8)

By Hideaki WAKAKO*) and Yoshitsugu TAKEI**)

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Abstract: Exact WKB analysis for instanton-type solutions of the degenerate third Painlevé equation of type (D_8) is discussed. Explicit connection formulas are obtained through computations of the monodromy data of the underlying linear equations.

Key words: Exact WKB analysis, connection formula, third Painlevé equation.

1. Introduction. In this paper we discuss the exact WKB analysis for instanton-type solutions (i.e., 2-parameter formal solutions) of the following degenerate third Painlevé equation of type (D_8) with a large parameter η:

$$(P)\qquad \frac{d^2q}{dt^2} = \frac{1}{q}\left(\frac{dq}{dt}\right)^2 - \frac{1}{t}\left(\frac{dq}{dt}\right) + \eta^2\left(\frac{q^2}{t^2} - \frac{1}{t}\right).$$

Exact WKB analysis for instanton-type solutions of Painlevé equations (P_J) ($J = \mathrm{I}, \ldots, \mathrm{VI}$) with a large parameter has been developed in [3, 1, 4, 9] etc. On the other hand, since the work of Sakai [8] on geometrical classification of the space of initial conditions of (P_J), it is considered to be natural to distinguish the degenerate third Painlevé equations of type (D_7) and (D_8) from the generic third Painlevé equation (P_{III}): Separately from (P_{III}), several important properties (such as τ-functions, irreducibility etc.) and asymptotics of solutions of the degenerate third Painlevé equations are studied in [7] and [5], respectively. The above equation (P) is obtained from an equation equivalent to the most degenerate third Painlevé equation of type (D_8) by introducing a large parameter η through an appropriate scaling of variables (or through the degeneration from (P_{III}); cf. [10]). From the viewpoint of exact WKB analysis (P) is also very peculiar: There is no turning point of (P) in the sense of [3] while it has two singular points $t = 0$ and ∞. In particular, $t = 0$ can be regarded as a non-linear analogue of a "singular point of simple pole type" (i.e., a singular point which also plays the role of turning points) of a second order linear differential equation discussed in [6]. In fact, as we will show in §3 below, a Stokes curve of (P) emanates from $t = 0$. The purpose of this paper is to discuss the Stokes phenomenon and connection formula for instanton-type solutions of (P) on a Stokes curve emanating from $t = 0$.

To analyze the Stokes phenomena, we make full use of the well-known fact that Painlevé equations govern the isomonodromic deformations of underlying systems of linear differential equations in the sense of [2]. In the case of (P) it is formulated as follows (cf. [7, §3]): Let (SL) and (D) denote the following linear differential equations, respectively.

$$(SL)\qquad \left(-\frac{\partial^2}{\partial x^2} + \eta^2 Q\right)\psi = 0,$$

$$(D)\qquad \frac{\partial \psi}{\partial t} = A\frac{\partial \psi}{\partial x} - \frac{1}{2}\frac{\partial A}{\partial x}\psi,$$

where

$$\begin{aligned} Q = \frac{tK}{x^2} + \frac{1}{2x} + \frac{t}{2x^3} - \eta^{-1}\frac{qp}{x(x-q)} \\ - \eta^{-2}\frac{1}{x(x-q)} + \eta^{-2}\frac{3}{4(x-q)^2}, \\ tK = q^2p^2 - \left(\frac{q}{2} + \frac{t}{2q}\right) + \eta^{-1}qp, \quad A = \frac{xq}{t(x-q)}. \end{aligned}$$

Then the compatibility condition of (SL) and (D) is described by the Hamiltonian system

$$(H)\qquad \frac{dq}{dt} = \eta\frac{\partial K}{\partial p}, \quad \frac{dp}{dt} = -\eta\frac{\partial K}{\partial q}$$

which is equivalent to (P). Consequently the monodromy data of (SL) are preserved (i.e., not depending on t) if a solution of (H) is substituted into the coefficients of (SL). In this paper, following the argument of [9] where the connection formula for (P_{I})

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*) Kaiyo Academy, 3–12–1 Kaiyo-cho, Gamagori-shi, Aichi 443–8588, Japan.

**) Research Institute for Mathematical Sciences, Kyoto University, Kitashirakawa Oiwake-cho, Sakyo-ku, Kyoto 606–8502, Japan.

is discussed, we explicitly compute the monodromy data of (SL) to write down the connection formula for (P).

2. Instanton-type solutions of (*P*). First of all, we introduce instanton-type solutions of (P).

We can readily see that $q = \pm\sqrt{t}$ and $(q,p) = \pm(\sqrt{t}, -\eta^{-1}/(4\sqrt{t}))$ respectively satisfy (P) and (H). As these solutions contain no free parameters, they are called 0-parameter solutions. In what follows we adopt

$$q^{(0)} = \sqrt{t}, \quad (q^{(0)}, p^{(0)}) = \left(\sqrt{t}, -\eta^{-1}\frac{1}{4\sqrt{t}}\right) \tag{1}$$

as 0-parameter solutions of (P) and (H).

Instanton-type formal solutions $q(t,\eta;\alpha,\beta)$ containing 2 free parameters (α,β) (or "2-parameter solutions" for short) are then constructed through the multiple-scale analysis. See [1, §1] for details. In particular, similarly to the case of (P_{I}) (cf. [9, §1]), we can construct the following 2-parameter solutions of (P) with homogeneity:

$$q(t,\eta;\alpha,\beta) = \sqrt{t} + \eta^{-1/2}\sum_{n=0}^{\infty}\eta^{-n/2}L_{n/2}(t,\eta), \tag{2}$$

where $L_0 = L_0(t,\eta)$ is given by

$$L_0 = t^{3/8}\left(\alpha(t^{1/4}\eta)^{\sqrt{2}\gamma}e^{\phi\eta} + \beta(t^{1/4}\eta)^{-\sqrt{2}\gamma}e^{-\phi\eta}\right)$$

with $\gamma = \alpha\beta$ and $\phi = 4\sqrt{2}t^{1/4}$, and $L_{n/2} = L_{n/2}(t,\eta)$ $(n \geq 1)$ is of the form

$$\sum_{k=0}^{n+1} c_{n+1-2k}^{(n/2)} t^{(3-n)/8}\left((t^{1/4}\eta)^{\sqrt{2}\gamma}e^{\phi\eta}\right)^{n+1-2k}$$

with $c_l^{(n/2)}$ being constants depending only on (α,β). Note that $q = q(t,\eta;\alpha,\beta)$ has homogeneity to the effect that $t^{-1/2}q$ is a formal series of one variable $t^{1/4}\eta$. Using the first equation of (H), i.e., $p = \eta^{-1}\{t(dq/dt) - q\}/(2q^2)$, we also obtain 2-parameter solutions $(q(t,\eta;\alpha,\beta), p(t,\eta;\alpha,\beta))$ of (H).

3. Stokes geometry of (*P*) and (*SL*). To define the Stokes geometry (i.e., turning points and Stokes curves) of (P), we consider its Frechét derivative at $q^{(0)} = \sqrt{t}$:

$$-\frac{d^2\varphi}{dt^2} + \eta^2\left(\frac{2}{t^{3/2}} - \eta^{-2}\frac{1}{4t^2}\right)\varphi = 0. \tag{3}$$

It is transformed by a change of variables $(t,\varphi) = (\tilde{t}^2, \tilde{t}^{1/2}\tilde{\varphi})$ into

$$-\frac{d^2\tilde{\varphi}}{d\tilde{t}^2} + \eta^2\left(\frac{8}{\tilde{t}} - \eta^{-2}\frac{1}{4\tilde{t}^2}\right)\tilde{\varphi} = 0. \tag{4}$$

Note that (4) has the same form as the equation discussed in [6]. As is proved in [6], $\tilde{t} = 0$ plays the role of a turning point of (4). Having this result in mind, we consider $\tilde{t} = 0$, i.e., $t = 0$ as a turning point of (P), though (P) has no ordinary turning point.

Definition 3.1. (i) We call $t = 0$ a turning point of (P).
(ii) A Stokes curve of (P) is by definition

$$\left\{t \in \mathbf{C} \mid \mathrm{Im}\int_0^t \sqrt{\frac{2}{t^{3/2}}}dt = \mathrm{Im}(4\sqrt{2}t^{1/4}) = 0\right\}. \tag{5}$$

By the definition (5) the Stokes curves of (P) are explicitly given by $\{t \in \mathbf{C} \mid \arg\sqrt{t} = 2n\pi \ (n \in \mathbf{Z})\}$.

We now study the relationship between the Stokes geometry of (P) and that of (SL). Here and in what follows we assume 2-parameter solutions $(q(t,\eta;\alpha,\beta), p(t,\eta;\alpha,\beta))$ are substituted into the coefficients of (SL) and (D). Then Q becomes an infinite series (in $\eta^{-1/2}$) of the form $Q = \sum_{n\geq 0}\eta^{-n/2}Q_{n/2}$ with

$$Q_0 = \frac{(x-\sqrt{t})^2}{2x^3} \quad \text{and} \quad Q_{1/2} \equiv 0.$$

Hence (SL) has only one turning point at $x = \sqrt{t}$, which is double. Furthermore, letting γ be a positively oriented circle $\{|x| = \sqrt{t}\}$ starting and ending at $x = \sqrt{t}$, we find

$$\int_\gamma \sqrt{Q_0}dx = \frac{1}{\sqrt{2}}\int_\gamma\left(\frac{1}{\sqrt{x}} - \frac{\sqrt{t}}{\sqrt{x^3}}\right)dx = -4\sqrt{2}t^{1/4}.$$

This implies

$$\mathrm{Im}\int_\gamma \sqrt{Q_0}dx = 0 \Longleftrightarrow \arg\sqrt{t} = 2n\pi \ (n \in \mathbf{Z}).$$

Thus we have

Proposition 3.2. (i) *(SL) has a unique turning point at $x = \sqrt{t}$, which is double.*
(ii) *When and only when $\arg\sqrt{t} = 2n\pi$ $(n \in \mathbf{Z})$, there exists a Stokes curve of (SL) that starts from $\sqrt{t}$, encircles $t = 0$ and returns to $\sqrt{t}$. It is the circle centered at the origin with radius $\sqrt{t}$ (cf. Fig.1).*

4. Canonical form of (*SL*) and (*D*) near the double turning point. In this section, as a preparation for computations of the monodromy data of (SL), we discuss the transformation of (SL) and (D) near the double turning point $x = \sqrt{t}$ into their canonical form.

We first introduce the following WKB solutions as fundamental systems of solutions of (SL).

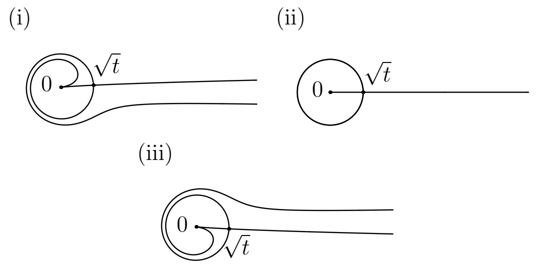

Fig. 1. Stokes curves of (SL) in the case of (i) $\arg\sqrt{t} > 0$, (ii) $\arg\sqrt{t} = 0$ and (iii) $\arg\sqrt{t} < 0$.

(6)
$$\psi_{\pm}^{(k)} = \frac{1}{\sqrt{S_{\mathrm{odd}}}} \exp\left(\pm\eta 2\sqrt{2}t^{1/4}\right) \times \exp\pm\left(\eta\int_{\sqrt{t}}^{x} S_{-1}dx + \int_{k}^{x}(S_{\mathrm{odd}} - \eta S_{-1})dx\right),$$

where $k = 0$ or ∞, $S = \sum_{n\geq -2}\eta^{-n/2}S_{n/2}$ is a formal power series solution of the Riccati equation $S^2 + (\partial S/\partial x) = \eta^2 Q$ associated with (SL) and S_{odd} denotes its odd part in the sense of [1, Def. 3.1]. Note that both WKB solutions (6) are well-defined since $S_{\mathrm{odd}} = \eta S_{-1} + \sum_{n\geq 0}\eta^{-n/2}S_{\mathrm{odd},n/2}$ satisfies that $x^{1/2}S_{\mathrm{odd},n/2}$ (resp., $x^{3/2}S_{\mathrm{odd},n/2}$) are holomorphic at $x = 0$ (resp., $x = \infty$) for $n \geq 0$. Here and in what follows we assume the branch of (6) is chosen so that $(x - \sqrt{t})^{1/2} > 0$ for $x > \sqrt{t}$ and $x^{-3/4} > 0$ for $x > 0$ may hold (in defining $\sqrt{S_{-1}}$) when $\arg\sqrt{t} = 0$. (As we are interested in Stokes phenomena for (P), we may assume that t lies near a Stokes curve $\arg\sqrt{t} = 0$ of (P).) The WKB solutions (6) then become single-valued in a cut plane indicated in Fig.2. We also have the following relation between $\psi_{\pm}^{(0)}$ and $\psi_{\pm}^{(\infty)}$:

$$(7) \qquad \psi_{\pm}^{(\infty)} = \left(\exp\pm\left(\pi i \operatorname{Res}_{x=\sqrt{t}} S_{\mathrm{odd}}\right)\right)\psi_{\pm}^{(0)}.$$

Now, using $(\partial/\partial t)S_{\mathrm{odd}} = (\partial/\partial x)(AS_{\mathrm{odd}})$ (cf. [1, (2.14)]), we can confirm the following

Proposition 4.1. *Both WKB solutions $\psi_{\pm}^{(0)}$ and $\psi_{\pm}^{(\infty)}$ satisfy (D).*

Furthermore, the WKB solutions (6) enjoy the following homogeneity property: Letting $\mathcal{H}$ be a scaling operator defined by $\mathcal{H} : (x, t, \eta) \mapsto (r^{-2}x, r^{-4}t, r\eta)$, we find (6) are homogeneous of degree -1 for $\mathcal{H}$ (i.e., $\psi_{\pm}^{(k)}(\mathcal{H}(x, t, \eta)) = r^{-1}\psi_{\pm}^{(k)}(x, t, \eta)$ hold).

To determine the connection formula for the WKB solutions (6) on Stokes curves of (SL) emanating from $x = \sqrt{t}$, we make use of the transformation theorem proved in [4] for Painlevé equations: Let (SL_{can}) and (D_{can}) denote the following equations,

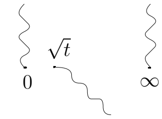

Fig. 2. x-plane with cuts. (Wiggly lines designate cuts.)

respectively.

$$(SL_{\mathrm{can}}) \qquad \left(-\frac{\partial^2}{\partial z^2} + \eta^2 Q_{\mathrm{can}}\right)\phi = 0,$$

$$(D_{\mathrm{can}}) \qquad \frac{\partial\phi}{\partial s} = A_{\mathrm{can}}\frac{\partial\phi}{\partial z} - \frac{1}{2}\frac{\partial A_{\mathrm{can}}}{\partial z}\phi,$$

where

$$Q_{\mathrm{can}} = 4z^2 + \eta^{-1}E + \frac{\eta^{-3/2}\rho}{z - \eta^{-1/2}\sigma} + \frac{3\eta^{-2}}{4(z - \eta^{-1/2}\sigma)^2},$$
$$E = \rho^2 - 4\sigma^2, \quad A_{\mathrm{can}} = \frac{1}{2(z - \eta^{-1/2}\sigma)}.$$

The compatibility condition of (SL_{can}) and (D_{can}) is given by the following Hamiltonian system:

$$(H_{\mathrm{can}}) \qquad \frac{\partial\rho}{\partial s} = -4\eta\sigma, \quad \frac{\partial\sigma}{\partial s} = -\eta\rho$$

(cf. [4, Prop. 2.1]). In what follows $(\rho_{\mathrm{can}}, \sigma_{\mathrm{can}})$ denotes a solution of (H_{can}) and $E_{\mathrm{can}} = \rho_{\mathrm{can}}^2 - 4\sigma_{\mathrm{can}}^2$, that is,

$$(8) \qquad \begin{cases} \sigma_{\mathrm{can}}(s, \eta) = A(\eta)e^{2\eta s} + B(\eta)e^{-2\eta s}, \\ \rho_{\mathrm{can}}(s, \eta) = -2A(\eta)e^{2\eta s} + 2B(\eta)e^{-2\eta s}, \\ E_{\mathrm{can}}(\eta) = -16A(\eta)B(\eta), \end{cases}$$

with $A(\eta) = \sum\eta^{-n/2}A_{n/2}$ and $B(\eta) = \sum\eta^{-n/2}B_{n/2}$ being formal power series with constant coefficients. Then the following theorem holds:

Theorem 4.2. *For any given 2-parameter (α, β) and a point t_0 in question, there exist a neighborhood V of t_0, a neighborhood U of $x = \sqrt{t_0}$, formal series $(A(\eta), B(\eta)) = (\sum\eta^{-n/2}A_{n/2}, \sum\eta^{-n/2}B_{n/2})$, and formal series $z(x, t, \eta) = \sum\eta^{-n/2}z_{n/2}(x, t)$ and $s(t, \eta) = \sum\eta^{-n/2}s_{n/2}(t)$ whose coefficients $z_{n/2}$ and $s_{n/2}$ are holomorphic on $U \times V$ and V respectively, so that the following holds: If $\phi(z, s, \eta)$ is a WKB solution of (SL_{can}) which also satisfies (D_{can}), then*

$$(9) \qquad \psi(x, t, \eta) = \left(\frac{\partial z}{\partial x}\right)^{-1/2}\phi(z(x, t, \eta), s(t, \eta), \eta)$$

satisfies both (SL) and (D).

For the proof see [4, Prop. 3.1]. In our case, by the same reasoning as that used in [9, §2.3] for the underlying linear equations of (P_{I}), we can verify that

$z(x,t,\eta)$ and $s(t,\eta)$ are homogeneous for $\mathcal{H}$ of degree $-1/2$ and -1 respectively. Furthermore, $A(\eta)$ and $B(\eta)$ can be taken so that

$$A(\eta) = 2^{-3/4}\alpha, \quad B(\eta) = 2^{-3/4}\beta \tag{10}$$

may hold and $E = E_{\mathrm{can}}(\eta)$ also satisfies

$$E = -4\sqrt{2}\alpha\beta = 4\,\mathrm{Res}_{x=\sqrt{t}}\, S_{\mathrm{odd}}. \tag{11}$$

See [9, §2.3] for the proof of (10) and (11).

As in [9, §2.3], we take the following WKB solutions of (SL_{can}):

$$\phi_{\pm} = \frac{1}{\sqrt{T_{\mathrm{odd}}}}(\eta^{1/2}z)^{\pm E/4} \times \exp \pm \left(\eta\int_0^z T_{-1}dz + \int_{\infty}^z \left(T_{\mathrm{odd}} - \eta T_{-1} - \frac{E}{4z}\right)dz\right),$$

where $T = \sum_{n\geq -2}\eta^{-n/2}T_{n/2}$ is a solution of the Riccati equation associated with (SL_{can}) and T_{odd} denotes its odd part. For the fundamental properties (such as the well-definedness) of $\phi_{\pm}$ we refer the reader to [9, §2.3] and only recall the following important properties here: $e^{\pm\eta s}\phi_{\pm}$ also satisfy (D_{can}) ([9, Lemma 2]) and further $\phi_{\pm}$ are homogeneous of degree $-1/4$ for a scaling operator $\tilde{\mathcal{H}} : (z,s,\eta) \mapsto (r^{-1/2}z, r^{-1}s, r\eta)$.

Between $\phi_{\pm}$ and (6) we have the following

Proposition 4.3.

$$\psi_{\pm}^{(k)} = C_{\pm}^{(k)}\left(\frac{\partial z}{\partial x}\right)^{-1/2}\phi_{\pm}(z(x,t,\eta), s(t,\eta), \eta) \tag{12}$$

($k = 0, \infty$) hold with

$$C_{\pm}^{(\infty)} = 2^{\mp 5E/16}e^{\pm\eta s(t,\eta)}, \quad C_{\pm}^{(0)} = e^{\mp\pi i E/4}C_{\pm}^{(\infty)}. \tag{13}$$

Using Theorem 4.2 and the homogeneity of $\psi_{\pm}^{(\infty)}$, $\phi_{\pm}$ and $(z(x,t,\eta), s(t,\eta))$, we can prove Proposition 4.3 by the same argument as in the proof of [9, Prop. 3]. Note that the second relation of (13) is an immediate consequence of (7) and (11).

Combining Proposition 4.3 and the connection formula for $\phi_{\pm}$ ([9, Prop. 4]), we obtain the following connection formulas for the WKB solutions (6): Here we label the Stokes curves and the Stokes regions near $x = \sqrt{t}$ as is indicated in Fig.3. We also use the notation $\psi_{\pm}^{(k),R}$ ($k = 0, \infty$) to denote the Borel sum of $\psi_{\pm}^{(k)}$ in a Stokes region R here and in what follows.

$$\begin{cases} \psi_{+}^{(k),R_{j-1}} = \psi_{+}^{(k),R_j} + \frac{C_{+}^{(k)}}{C_{-}^{(k)}}a_{j-1j}\psi_{-}^{(k),R_j} \\ \psi_{-}^{(k),R_{j-1}} = \psi_{-}^{(k),R_j} \end{cases} \tag{14}$$

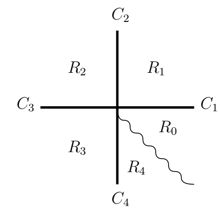

Fig. 3. Stokes curves and Stokes regions near $x = \sqrt{t}$.

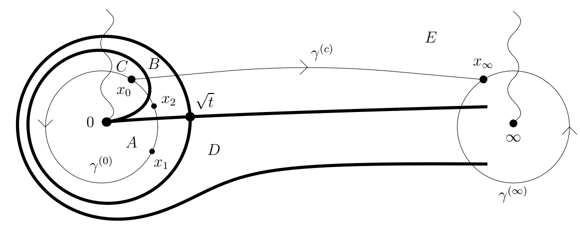

Fig. 4. Paths of analytic continuation $\gamma^{(0)}$, $\gamma^{(c)}$ and $\gamma^{(\infty)}$ and Stokes curves for $\arg\sqrt{t} > 0$. (A, B, C, ... designate the label of Stokes regions.)

on C_j for $j = 1, 3$ and

$$\begin{cases} \psi_{+}^{(k),R_{j-1}} = \psi_{+}^{(k),R_j} \\ \psi_{-}^{(k),R_{j-1}} = \psi_{-}^{(k),R_j} + \frac{C_{-}^{(k)}}{C_{+}^{(k)}}a_{j-1j}\psi_{+}^{(k),R_j} \end{cases} \tag{15}$$

on C_j for $j = 2, 4$, where $C_{\pm}^{(k)}$ ($k = 0, \infty$) are defined by (13) and a_{j-1j} are given as follows:

$$(-1)^{(j+1)/2}\frac{\rho + 2\sigma}{2}\frac{i\sqrt{2\pi}}{\Gamma(1-\frac{E}{4})}2^{-E/2}e^{(j-1)\pi i E/4} \tag{16}$$

for $j = 1, 3$ and

$$(-1)^{(j-2)/2}\frac{\rho - 2\sigma}{2}\frac{\sqrt{2\pi}}{\Gamma(1+\frac{E}{4})}2^{E/2}e^{(1-j)\pi i E/4} \tag{17}$$

for $j = 2, 4$ with $(\rho, \sigma) = (\rho_{\mathrm{can}}, \sigma_{\mathrm{can}})$, $E = -4\sqrt{2}\alpha\beta$.

5. Computation of the monodromy data of (SL). In this section, using the connection formulas (14) and (15), we explicitly compute the monodromy data of (SL).

First, we review the monodromy data of (SL) (cf. [2]). As in Fig.4, let us take base points x_0, x_∞ and paths of analytic continuation $\gamma^{(k)}$ ($k = 0, c, \infty$). Further, we take fundamental systems $\varphi_{\pm}^{k}$ of holomorphic solutions near x_k ($k = 0, \infty$). Then, according to [2, §2], the monodromy data of (SL) is given by the following set of matrices

$$\{M_0,\ M_c,\ M_\infty\},$$

where the matrices M_k ($k = 0, c, \infty$) are defined by

$$(\gamma_*^{(k)})(\varphi_+^k, \varphi_-^k) = (\varphi_+^k, \varphi_-^k)M_k \quad (\text{for } k = 0, \infty),$$
$$(\gamma_*^{(c)})(\varphi_+^0, \varphi_-^0) = (\varphi_+^\infty, \varphi_-^\infty)M_c.$$

Here and in what follows $\gamma_*(f)$ designates the analytic continuation of f along a path γ.

Remark 5.1. Since $x = 0$ and $x = \infty$ are irregular singular points (with Poincaré rank 1/2) of (SL), the monodromy matrices M_0 and M_∞ are expressed in terms of the (triangular) Stokes matrices S_0 and S_∞ as

$$M_0 = S_0 J,\ M_\infty = J S_\infty \text{ with } J = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.$$

Note that the matrix J appears as an effect of crossing a cut emanating from $x = 0$ or $x = \infty$. In the case of (SL) we can also confirm the following

$$-(M_0)^{-1} = (M_c)^{-1} M_\infty M_c. \tag{18}$$

Thanks to (18), if we write

$$S_0 = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix},\ S_\infty = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix},\ M_c = \begin{pmatrix} c & d \\ e & f \end{pmatrix},$$

we find that all monodromy data are determined once a and d are computed.

In what follows, we adopt the Borel sum of $\psi_\pm^{(k)}$ near x_k $(k = 0, \infty)$ as fundamental systems of solutions, i.e., $\varphi_\pm^0 = \psi_\pm^{(0),C}$ and $\varphi_\pm^\infty = \psi_\pm^{(\infty),E}$, and compute M_0, M_c when $\arg\sqrt{t} > 0$ and $\arg\sqrt{t} < 0$ respectively. To illustrate how the computations are done, we explain the computation of M_0 for $\arg\sqrt{t} > 0$ in details here.

First, we consider the analytic continuation from a point x_1 in Region A to a point x_2 in Region B (cf. Fig.4). It is described by the connection formula (14) for $j = 3$, that is,

$$(\psi_+^{(0),A}, \psi_-^{(0),A}) = (\psi_+^{(0),B}, \psi_-^{(0),B}) \begin{pmatrix} 1 & 0 \\ -\dfrac{C_+^{(0)}}{C_-^{(0)}} a_{23} & 0 \end{pmatrix}. \tag{19}$$

Second, we discuss the analytic continuation from x_2 to x_0. We divide this step of analytic continuation into the following three substeps; (i) from x_2 to x_D along γ_{BD}, (ii) from x_D to x_A across a Stokes curve emanating from $\sqrt{t}$, and (iii) from x_A to x_0 along γ_{AC} (cf. Fig.5). The substep (i) is described by

$$(\gamma_{BD})_*(\psi_+^{(0),B}, \psi_-^{(0),B}) = (\psi_+^{(0),D}, \psi_-^{(0),D}) \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix},$$

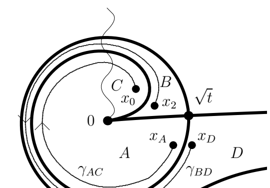

Fig. 5. Paths γ_{BD} and γ_{AC}.

the substep (ii) is described by the connection formula (15) for $j = 4$:

$$(\psi_+^{(0),D}, \psi_-^{(0),D}) = (\psi_+^{(0),A}, \psi_-^{(0),A}) \begin{pmatrix} 1 & -\dfrac{C_-^{(0)}}{C_+^{(0)}} a_{34} \\ 0 & 1 \end{pmatrix},$$

and the substep (iii) is described by

$$(\gamma_{AC})_*(\psi_+^{(0),A}, \psi_-^{(0),A}) = (\psi_+^{(0),C}, \psi_-^{(0),C}) \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{20}$$

Combining these three substeps, we obtain

$$(\psi_+^{(0),B}, \psi_-^{(0),B}) = (\psi_+^{(0),C}, \psi_-^{(0),C}) \begin{pmatrix} 1 & 0 \\ -\dfrac{C_-^{(0)}}{C_+^{(0)}} a_{34} & 1 \end{pmatrix} \tag{21}$$

for the analytic continuation from x_2 to x_0. Finally, (20) also entails

$$(\gamma_{AC}^{-1})_*(\psi_+^{(0),C}, \psi_-^{(0),C}) = (\psi_+^{(0),A}, \psi_-^{(0),A}) \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \tag{22}$$

which describes the analytic continuation from x_0 to x_1. We thus conclude from (19), (21) and (22) that

$$\gamma_*^{(0)}(\psi_+^{(0),C}, \psi_-^{(0),C}) = (\psi_+^{(0),C}, \psi_-^{(0),C}) M_0 \tag{23}$$

with

$$M_0 = \begin{pmatrix} 1 & 0 \\ -\dfrac{C_+^{(0)}}{C_-^{(0)}} a_{23} - \dfrac{C_-^{(0)}}{C_+^{(0)}} a_{34} & 1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \tag{24}$$

for $\arg\sqrt{t} > 0$.

The computation of M_c for $\arg\sqrt{t} > 0$ and that of M_0 and M_c for $\arg\sqrt{t} < 0$ can be done in a similar manner. Here, omitting the details of computations, we give only the consequence of them:

$$M_c = \begin{pmatrix} \dfrac{C_+^{(0)}}{C_+^{(\infty)}} - \dfrac{(C_-^{(0)})^2}{C_+^{(0)} C_+^{(\infty)}} a_{12} a_{34} & -\dfrac{C_-^{(0)}}{C_+^{(\infty)}} a_{12} \\ \dfrac{(C_-^{(0)})^2}{C_+^{(0)} C_-^{(\infty)}} a_{34} & \dfrac{C_-^{(0)}}{C_-^{(\infty)}} \end{pmatrix} \tag{25}$$

for $\arg\sqrt{t} > 0$,

$$(26)\quad M_0 = \begin{pmatrix} 1 & 0 \\ -\frac{C_+^{(0)}}{C_-^{(0)}}a_{23} - \frac{C_-^{(0)}}{C_+^{(0)}}a_{12} & 1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix},$$

(27)

$$M_c = \begin{pmatrix} \frac{C_+^{(0)}}{C_+^{(\infty)}} & -\frac{C_-^{(0)}}{C_+^{(\infty)}}a_{12} \\ \frac{(C_-^{(0)})^2}{C_+^{(0)}C_-^{(\infty)}}a_{34} & \frac{C_-^{(0)}}{C_-^{(\infty)}} - \frac{(C_-^{(0)})^3}{(C_+^{(0)})^2C_-^{(\infty)}}a_{12}a_{34} \end{pmatrix}$$

for $\arg\sqrt{t} < 0$.

Using these results together with (8), (10), (11), (13), (16) and (17), we thus obtain the following formulas (with $E = -4\sqrt{2}\alpha\beta$) for the relevant monodromy data a and d explained in Remark 5.1.

(When $\arg\sqrt{t} > 0$)

$$(28)\quad \begin{aligned} a =& -2^{-9E/8+1/4}\frac{i\sqrt{2\pi}\beta}{\Gamma(-\frac{E}{4}+1)} \\ & -2^{9E/8+1/4}e^{-i\pi E/4}\frac{\sqrt{2\pi}\alpha}{\Gamma(\frac{E}{4}+1)}, \\ d =& 2^{9E/8+1/4}\frac{\sqrt{2\pi}\alpha}{\Gamma(\frac{E}{4}+1)}. \end{aligned}$$

(When $\arg\sqrt{t} < 0$)

$$(29)\quad \begin{aligned} a =& -2^{-9E/8+1/4}\frac{i\sqrt{2\pi}\beta}{\Gamma(-\frac{E}{4}+1)} \\ & +2^{9E/8+1/4}e^{i\pi E/4}\frac{\sqrt{2\pi}\alpha}{\Gamma(\frac{E}{4}+1)}, \\ d =& 2^{9E/8+1/4}\frac{\sqrt{2\pi}\alpha}{\Gamma(\frac{E}{4}+1)}. \end{aligned}$$

6. Connection formula for 2-parameter instanton-type solutions of (*P*). Finally we discuss the connection formula for 2-parameter instanton-type solutions of (P).

Let us now suppose that a 2-parameter solution $q(t,\eta;\alpha,\beta)$ in $\{t; \arg\sqrt{t} > 0\}$ and a 2-parameter solution $q(t,\eta;\tilde{\alpha},\tilde{\beta})$ in $\{t; \arg\sqrt{t} < 0\}$ may represent the same holomorphic solution of (P). Then, thanks to the result of [2], the corresponding monodromy data of (SL) for $\arg\sqrt{t} > 0$ should coincide with that for $\arg\sqrt{t} < 0$. Since the monodromy data is explicitly given by (28) and (29), we thus conclude (α,β) and $(\tilde{\alpha},\tilde{\beta})$ should satisfy

$$(30)\quad \begin{aligned} & 2^{-9E/8}\frac{i\beta}{\Gamma(-\frac{E}{4}+1)} + 2^{9E/8}e^{-i\pi E/4}\frac{\alpha}{\Gamma(\frac{E}{4}+1)} \\ & = 2^{-9\tilde{E}/8}\frac{i\tilde{\beta}}{\Gamma(-\frac{\tilde{E}}{4}+1)} - 2^{9\tilde{E}/8}e^{i\pi\tilde{E}/4}\frac{\tilde{\alpha}}{\Gamma(\frac{\tilde{E}}{4}+1)}, \\ & 2^{9E/8}\frac{\alpha}{\Gamma(\frac{E}{4}+1)} = 2^{9\tilde{E}/8}\frac{\tilde{\alpha}}{\Gamma(\frac{\tilde{E}}{4}+1)} \end{aligned}$$

with $E = -4\sqrt{2}\alpha\beta$ and $\tilde{E} = -4\sqrt{2}\tilde{\alpha}\tilde{\beta}$. By (30) we find that (α,β) and $(\tilde{\alpha},\tilde{\beta})$ are different in general. This is the Stokes phenomenon for $q(t,\eta;\alpha,\beta)$ and (30) gives their connection formula.

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