A generalization of Nochka weight function

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Abstract: For any subset, which is not always finite, of $\boldsymbol{C}^{n+1}-\{\mathbf{0}\}$ in subgeneral position, we introduce a weight function and a constant like as Nochka weight function and Nochka constant which was introduced for a finite subset of $\boldsymbol{C}^{n+1}-\{\mathbf{0}\}$ in subgeneral position.

Key words: Nochka weight function; Nochka constant; holomorphic curve.

1. Introduction. About twenty four years ago, E. I. Nochka [4] proved the conjecture of H. Cartan given in [1] by introducing a weight function and a constant for a finite subset of $\boldsymbol{C}^{n+1}-\{\mathbf{0}\}$.

Let N, n be integers satisfying $N\geq n\geq 1$ and X be any subset of $\boldsymbol{C}^{n+1}-\{\mathbf{0}\}$ in N–subgeneral position. For a finite subset P of X, we denote by $V(P)$ the subspace of $\boldsymbol{C}^{n+1}$ generated by elements of P and by $d(P)$ the dimension of $V(P)$. We put $\mathcal{O}=\{P\subset X\mid 0<\#P\leq N+1\}$.

When $2N-n+1\leq\#X<\infty$, it is known (see [2,3,5,6]) that there exist a constant θ and a function $\omega: X\to(0,1]$ with the following properties:

Theorem 1.A.

(1.a) *For any* $\boldsymbol{a}\in X$, $0<\theta\omega(\boldsymbol{a})\leq 1$;

(1.b) $\#X-(2N-n+1)=\theta(\sum_{\boldsymbol{a}\in X}\omega(\boldsymbol{a})-n-1)$;

(1.c) $(N+1)/(n+1)\leq\theta\leq(2N-n+1)/(n+1)$;

(1.d) *For any* $P\in\mathcal{O}$, $\sum_{\boldsymbol{a}\in P}\omega(\boldsymbol{a})\leq d(P)$.

We call ω the Nochka weight function and θ the Nochka constant. E. I. Nochka [4] succeeded in solving the Cartan conjecture with these notions. We used them to obtain some results on holomorphic curves extremal for the defect relation [7,8]. But, it is incovenient to apply them to holomorphic curves with an infinite number of defects since the weight function is defined only for a finite set. We would like to delete the condition "$\#X<\infty$" in Theorem 1.A. To that end, we shall generalize ω and θ to any subset of $\boldsymbol{C}^{n+1}-\{\mathbf{0}\}$ in N–subgeneral position to obtain a new weight function and a constant satisfying properties like those of Theorem 1.A.

We can apply them to the value distribution theory of holomorphic curves with an infinite number of defects directly. Applications will appear elsewhere.

2. Preliminaries. Let N, n, X etc. be as in Section 1. From now on throughout the paper $\#X$ is not always finite.

Lemma 2.1 ([3; p.68]). *For* $S_1, S_2\in\mathcal{O}$,
$d(S_1\cup S_2)+d(S_1\cap S_2)\leq d(S_1)+d(S_2)$.

Lemma 2.2 ([3; p.68]). *For* $R\subset S$ $(R,S\in\mathcal{O})$,
$\#R-d(R)\leq\#S-d(S)\leq N-n$.

For $R\subsetneqq S$ $(R,S\in\mathcal{O})$, we put
$\Lambda(R;S)=(d(S)-d(R))/(\#S-\#R)$.

Then, by Lemma 2.2 we have the following

Proposition 2.1 ([3; p.67]). $0\leq\Lambda(R;S)\leq 1$.

Lemma 2.3. $\#\{d(S)/\#S\mid S\in\mathcal{O}\}$ *is finite.*

Proof. We have only to prove this lemma when $\#X$ is not finite. For any $S\in\mathcal{O}$,

$$(1)\quad 1\leq d(S)\leq n+1 \quad\text{and}\quad 1\leq\#S\leq N+1.$$

Further, from Lemma 2.2, $\#S-d(S)\leq N-n$, which reduces to the inequality

$$(2)\qquad \frac{1}{N-n+1}\leq\frac{d(S)}{N-n+d(S)}\leq\frac{d(S)}{\#S}\leq 1.$$

From (1) and (2), the number $d(S)/\#S$ can attain at most $(n+1)(N+1)$ rational numbers between 1 and $1/(N-n+1)$. □

From this lemma we can give the following definition.

Definition 2.1. $\lambda=\min_{S\in\mathcal{O}} d(S)/\#S$.

Proposition 2.2. *It holds that*
$1/(N-n+1)\leq\lambda\leq(n+1)/(N+1)$.

Proof. From (2), we have $1/(N-n+1)\leq\lambda$. On the other hand, for $S\in\mathcal{O}$ such that $\#S=N+1$, $d(S)=n+1$ by the definition of N-subgeneral position and we have $\lambda\leq(n+1)/(N+1)$. □

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Lemma 2.4. *For a fixed $R \in \mathcal{O}$,*
$\#\{\Lambda(R;S) \mid R \subsetneqq S \in \mathcal{O}\} < \infty$.

Proof. We have only to prove this lemma when $\#X$ is not finite. As $0 \leq d(S) - d(R) \leq n$ and $1 \leq \#S - \#R = \#(S-R) \leq N$, $\Lambda(R;S)$ can attain at most $N(n+1)$ rational numbers between 0 and 1. □

Proposition 2.3. (I) *When $\lambda \geq (n+1)/(2N-n+1)$, for any $S \in \mathcal{O}$ it holds that*
$d(S)/\#S \geq (n+1)/(2N-n+1)$.

(II) *When $\lambda < (n+1)/(2N-n+1)$, there exist an integer p $(1 \leq p < (n+1)/2)$ and a subfamily $\{T_i \mid 1 \leq i \leq p\}$ of $\mathcal{O}$ satisfying the conditions*:

(i) $\phi = T_0 \subsetneqq T_1 \subsetneqq \cdots \subsetneqq T_p$, $d(T_p) < (n+1)/2$;

(ii) $\Lambda(T_0;T_1) < \Lambda(T_1;T_2) < \cdots < \Lambda(T_{p-1};T_p)$
$< (n+1-d(T_p))/(2N-n+1-\#T_p)$;

(iii) *When $1 \leq i \leq p$, for any $U \in \mathcal{O}$ such that $T_{i-1} \subsetneqq U$, if $d(T_{i-1}) < d(U)$, then*

(a) $\Lambda(T_{i-1};T_i) \leq \Lambda(T_{i-1};U)$ *and*

(b) $\Lambda(T_{i-1};T_i) = \Lambda(T_{i-1};U)$ *only if* $U \subseteqq T_i$;

(iv) *For any $U \in \mathcal{O}$ such that $T_p \subsetneqq U$, if $d(T_p) < d(U)$, then*
$\Lambda(T_p;U) \geq (n+1-d(T_p))/(2N-n+1-\#T_p)$.

Proof. (I) This is trivial by the definition of λ.

(II) Note that $N > n \geq 2$ by Definition 2.1 and Proposition 2.2. We put $T_0 = \phi$ and
$\mathcal{O}(\lambda) = \{S \in \mathcal{O} \mid d(S)/\#S = \lambda\}$.

Step 1. (a_1) $\mathcal{O}(\lambda)$ is not empty.

(b_1) Let $S \in \mathcal{O}(\lambda)$. Then,

$$(3)\quad d(S) < (n+1)/2 \quad \text{and} \quad \#S < (2N-n+1)/2.$$

In fact, as $d(S)/\#S = \lambda < (n+1)/(2N-n+1)$, we obtain (3) by Lemma 2.2.

(c_1) If $S_1, S_2 \in \mathcal{O}(\lambda)$, then $S_1 \cup S_2 \in \mathcal{O}(\lambda)$.

In fact, from Lemma 2.1 and (b_1) we obtain the inequality

$$d(S_1 \cup S_2) + d(S_1 \cap S_2) \leq d(S_1) + d(S_2) < (n+1)/2 + (n+1)/2 = n+1,$$

so that $d(S_1 \cup S_2) \leq n$. This implies that $\#(S_1 \cup S_2) \leq N$ and so $S_1 \cup S_2 \in \mathcal{O}$. Next, by the definition of λ and by Lemma 2.1,

$$\lambda \leq \frac{d(S_1 \cup S_2)}{\#(S_1 \cup S_2)} \leq \frac{d(S_1) + d(S_2) - d(S_1 \cap S_2)}{\#S_1 + \#S_2 - \#(S_1 \cap S_2)} = (*)$$

and by using the inequality $\lambda\#(S_1 \cap S_2) \leq d(S_1 \cap S_2)$ we have

$$(*) \leq \frac{\lambda(\#S_1 + \#S_2 - \#(S_1 \cap S_2))}{\#S_1 + \#S_2 - \#(S_1 \cap S_2)} = \lambda.$$

We obtain that $d(S_1 \cup S_2)/\#(S_1 \cup S_2) = \lambda$, which means that $S_1 \cup S_2 \in \mathcal{O}(\lambda)$.

(d_1) $\#\mathcal{O}(\lambda)$ is finite.

We have only to prove (d_1) when $\#X$ is not finite. Suppose to the contrary that $\#\mathcal{O}(\lambda) = \infty$. Then, $\mathcal{O}(\lambda) \supset \{S_1, S_2, \cdots, S_i, \cdots\}$, $S_i \neq S_j$ if $i \neq j$ and $\#\{\bigcup_{i=1}^{\infty} S_i\} = \infty$.

There exists an integer ν satisfying

$$(4)\qquad N+1 < \#\{\cup_{i=1}^{\nu} S_i\}.$$

On the other hand, $\cup_{i=1}^{\nu} S_i \in \mathcal{O}(\lambda)$ by (c_1) and so by (b_1)

$$(5)\qquad \#\{\cup_{i=1}^{\nu} S_i\} < (2N-n+1)/2.$$

From (4) and (5) we obtain that $n+1 < 0$, which is absurd. This imples that $\#\mathcal{O}(\lambda)$ must be finite.

(e_1) We put $T_1 = \bigcup_{S \in \mathcal{O}(\lambda)} S$. Then $T_1 \in \mathcal{O}(\lambda)$ from (c_1), (d_1), and if $S \in \mathcal{O}(\lambda)$, then $S \subset T_1$.

Moreover, T_1 satisfies the following conditions ((i), (ii), (iii) of Proposition 2.3(II) for $p = 1$):

(i_1) $\phi = T_0 \subsetneqq T_1$, $d(T_1) < (n+1)/2$;

(ii_1)
$\Lambda(T_0;T_1) < (n+1-d(T_1))/(2N-n+1-\#T_1)$;

(iii_1) For any $U \in \mathcal{O}$, (a) $\Lambda(T_0;T_1) \leq \Lambda(T_0;U)$ and (b) $\Lambda(T_0;T_1) = \Lambda(T_0;U)$ only if $U \subseteqq T_1$.

In fact, (i_1) is trivial by (a_1), (e_1) and (b_1).

(ii_1) As $d(T_1)/\#T_1 = \lambda < (n+1)/(2N-n+1)$, we obtain (ii_1) easily.

(iii_1) $\Lambda(T_0;T_1) = \lambda \leq d(U)/\#U$ for any $U \in \mathcal{O}$ by the definition of λ and T_1. If $U \in \mathcal{O}(\lambda)$ then $U \subseteqq T_1$ by (e_1).

Next, we put
$\mathcal{O}_1 = \{S \in \mathcal{O} \mid T_1 \subset S,\ d(T_1) < d(S)\}$.

(a_2) $\mathcal{O}_1$ is not empty.

In fact, as $d(T_1) < (n+1)/2$, any S such that $T_1 \subset S \in \mathcal{O}$ and $\#S = N+1$ belongs to $\mathcal{O}_1$.

(b_2) $\#\{\Lambda(T_1;S) \mid S \in \mathcal{O}_1\}$ is finite.

This follows from Lemma 2.4.

Here, we put $\lambda_1 = \min_{S \in \mathcal{O}_1} \Lambda(T_1;S)$. Then,

(c_2) $\lambda < \lambda_1$.

We prove this inequality. For any $S \in \mathcal{O}_1$, we have

$$(6)\qquad d(T_1)/\#T_1 < d(S)/\#S.$$

In fact, by (iii_1)

$$(7)\qquad d(T_1)/\#T_1 \leq d(S)/\#S$$

and if the equality holds in (7), $S \subseteqq T_1$, which is absurd. We obtain (6) and from which we have the inequality

$$\lambda = \frac{d(T_1)}{\#T_1} < \frac{d(S)}{\#S} < \frac{d(S) - d(T_1)}{\#S - \#T_1}$$

for any $S \in \mathcal{O}_1$, so that we obtain (c_2) due to (b_2).

When $\lambda_1 \geq (n+1-d(T_1))/(2N-n+1-\#T_1)$.

(iv_1) For any $U \in \mathcal{O}_1$,
$\Lambda(T_1; U) \geq (n+1-d(T_1))/(2N-n+1-\#T_1)$.

This means that our proposition holds for $p = 1$ and T_1.

Step 2. When $\lambda_1 < \frac{n+1-d(T_1)}{2N-n+1-\#T_1}$.

Suppose that there exist the sets $T_1, \cdots, T_i \in \mathcal{O}$ satisfying the following conditions:

(i_2) $\phi = T_0 \subsetneqq T_1 \subsetneqq \cdots \subsetneqq T_i$, $d(T_i) < (n+1)/2$;

(ii_2) $\Lambda(T_0; T_1) < \Lambda(T_1; T_2) < \cdots < \Lambda(T_{i-1}; T_i) < (n+1-d(T_i))/(2N-n+1-\#T_i)$;

(iii_2) When $1 \leq k \leq i$, for any $U \in \mathcal{O}$ such that $T_{k-1} \subsetneqq U$, if $d(T_{k-1}) < d(U)$, then

(a) $\Lambda(T_{k-1}; T_k) \leq \Lambda(T_{k-1}; U)$ and

(b) $\Lambda(T_{k-1}; T_k) = \Lambda(T_{k-1}; U)$ only if $U \subseteqq T_k$.

Note that from (ii_2) we obtain the inequality:

$$(8) \qquad \frac{d(T_1)}{\#T_1} < \frac{d(T_2)}{\#T_2} < \cdots < \frac{d(T_i)}{\#T_i} < \frac{n+1}{2N-n+1} < \frac{n+1-d(T_i)}{2N-n+1-\#T_i}$$

when $i \geq 2$.

We put $\mathcal{O}_0 = \mathcal{O}$ and for $1 \leq k \leq i$

$$\mathcal{O}_k = \{S \in \mathcal{O} \mid T_k \subset S,\ d(T_k) < d(S)\}.$$

We note that $\mathcal{O} \supset \mathcal{O}_1 \supset \cdots \supset \mathcal{O}_i$. Then, as in the case of $\mathcal{O}_1$, for $2 \leq i$ we have the following

(a_3) $\mathcal{O}_i$ is not empty;

(b_3) $\#\{\Lambda(T_i; S) \mid S \in \mathcal{O}_i\}$ is finite.

We put $\lambda_i = \min_{S \in \mathcal{O}_i} \Lambda(T_i; S)$. Then, as in ($c_2$) we have the following inequality.

(c_3) $(d(T_i) - d(T_{i-1}))/(\#T_i - \#T_{i-1}) < \lambda_i$.

In fact, for any $S \in \mathcal{O}_i$ we have the inequality $\Lambda(T_{i-1}; T_i) < \Lambda(T_{i-1}; S)$ from (iii_2), so that we have the inequality

$$\frac{d(T_i) - d(T_{i-1})}{\#T_i - \#T_{i-1}} < \frac{d(S) - d(T_{i-1})}{\#S - \#T_{i-1}} < \frac{d(S) - d(T_i)}{\#S - \#T_i},$$

from which we obtain the inequality (c_3).

Now, suppose that

$$(9) \quad \lambda_i < (n+1-d(T_i))/(2N-n+1-\#T_i).$$

Put $\mathcal{O}_i(\lambda_i) = \{S \in \mathcal{O}_i \mid \Lambda(T_i; S) = \lambda_i\}$. Then,

(a_4) $\mathcal{O}_i(\lambda_i)$ is not empty;

(b_4) For any $S \in \mathcal{O}_i(\lambda_i)$, $d(S) < (n+1)/2$ and $\#S < (2N-n+1)/2$.

In fact, from (9) we have $d(S) \leq n$ and $\#S \leq N$ so that from (ii_2), (c_3) and (9) we obtain the inequality

$$\frac{d(T_i)}{\#T_i} < \frac{d(S)}{\#S} < \frac{d(S)-d(T_i)}{\#S-\#T_i} < \frac{n+1-d(T_i)}{2N-n+1-\#T_i} < \frac{n+1-d(S)}{2N-n+1-\#S},$$

and so from the inequality

$d(S)/\#S < (n+1-d(S))/(2N-n+1-\#S)$

we obtain that $d(S)/\#S < (n+1)/(2N-n+1)$. By using Lemma 2.2, we obtain (b_4) as in (b_1).

(c_4) If $S_1,\ S_2 \in \mathcal{O}_i(\lambda_i)$, then $S_1 \cup S_2 \in \mathcal{O}_i(\lambda_i)$ (see [3; p.70]).

To prove (c_4), we first prove that

$$(10) \qquad S_1 \cup S_2 \in \mathcal{O}_i.$$

In fact, as

$$\lambda_i = \frac{d(S_1) - d(T_i)}{\#S_1 - \#T_i} = \frac{d(S_2) - d(T_i)}{\#S_2 - \#T_i},$$

by using Lemma 2.2 we obtain the inequality

$$\begin{aligned} d(S_1) + d(S_2) - 2d(T_i) &= \lambda_i(\#S_1 + \#S_2 - 2\#T_i) \\ &\leq \lambda_i(d(S_1) + N - n + d(S_2) \\ &\quad + N - n - 2\#T_i) \\ &= \lambda_i(d(S_1) + d(S_2) - 2d(T_i)) \\ &\quad + 2\lambda_i(N - n + d(T_i) - \#T_i), \end{aligned}$$

so that as $\lambda_i < 1$

$$\begin{aligned} &d(S_1) + d(S_2) - 2d(T_i) \\ &\quad \leq \tfrac{2\lambda_i}{1-\lambda_i}(N - n + d(T_i) - \#T_i) = (*). \end{aligned}$$

Here, we have the inequality

$$\begin{aligned} 1 - \lambda_i &> 1 - \frac{n+1-d(T_i)}{2N-n+1-\#T_i} \\ &= \frac{2N - 2n + d(T_i) - \#T_i}{2N - n + 1 - \#T_i}. \end{aligned}$$

By using this inequality and (9) we have

$$(*) < \lambda_i(2N - n + 1 - \#T_i) < n + 1 - d(T_i)$$

since $d(T_i) - \#T_i \leq 0$. We obtain the inequality $d(S_1) + d(S_2) - d(T_i) < n+1$, so that by Lemma 2.1 we have the inequality

$$\begin{aligned} d(S_1 \cup S_2) &\leq d(S_1) + d(S_2) - d(S_1 \cap S_2) \\ &\leq d(S_1) + d(S_2) - d(T_i) < n + 1 \end{aligned}$$

since $S_1 \cap S_2 \supset T_i$, which implies that $\#(S_1 \cup S_2) \leq N$ and we have $S_1 \cup S_2 \in \mathcal{O}$. Further, as $d(T_i) < d(S_1) \leq d(S_1 \cup S_2)$, we have (10).

Next, we prove the following inequality.

(11) $\lambda_i(\#(S_1 \cap S_2) - \#T_i) \leq d(S_1 \cap S_2) - d(T_i)$.

As this inequality is trivial when $\#(S_1 \cap S_2) - \#T_i = 0$, we prove it when $\#(S_1 \cap S_2) - \#T_i > 0$. First we prove that

(12) $$d(T_i) < d(S_1 \cap S_2).$$

Suppose to the contrary that $d(T_i) = d(S_1 \cap S_2)$. Then, as $T_i \subsetneqq S_1 \cap S_2 \in \mathcal{O}_{i-1}$ we have the inequality

$$\min_{S \in \mathcal{O}_{i-1}} \Lambda(T_{i-1}; S) = \Lambda(T_{i-1}; T_i) > \Lambda(T_{i-1}; S_1 \cap S_2).$$

This is a contradiction (see (iii$_2$)). We obtain (12) and $S_1 \cap S_2 \in \mathcal{O}_i$. By the definition of λ_i we have the inequality $\lambda_i \leq \Lambda(T_i; S_1 \cap S_2)$. This means that the inequality (11) holds.

Finally, we prove that $S_1 \cup S_2 \in \mathcal{O}_i(\lambda_i)$. By Lemma 2.1 and by (11)

$$\begin{aligned}\lambda_i &\leq \Lambda(T_i; S_1 \cup S_2) \\ &\leq \frac{d(S_1) + d(S_2) - d(S_1 \cap S_2) - d(T_i)}{\#S_1 + \#S_2 - \#(S_1 \cap S_2) - \#T_i} \leq \lambda_i\end{aligned}$$

since we obtain the following inequality from (11):

$$\begin{aligned}d(S_1) &+ d(S_2) - d(S_1 \cap S_2) - d(T_i) \\ &= d(S_1) - d(T_i) + d(S_2) - d(T_i) \\ &\quad - (d(S_1 \cap S_2) - d(T_i)) \\ &\leq \lambda_i(\#S_1 - \#T_i + \#S_2 - \#T_i \\ &\quad - (\#(S_1 \cap S_2) - \#T_i)) \\ &= \lambda_i(\#S_1 + \#S_2 - \#(S_1 \cap S_2) - \#T_i).\end{aligned}$$

Namely, we have that $\Lambda(T_i; S_1 \cup S_2) = \lambda_i$. This means that $S_1 \cup S_2 \in \mathcal{O}_i(\lambda_i)$.

As in Step 1 (d$_1$), we obtain the following

(d$_4$) $\#\mathcal{O}_i(\lambda_i)$ is finite.

(e$_4$) We put $T_{i+1} = \bigcup_{S \in \mathcal{O}_i(\lambda_i)} S$. Then $T_{i+1} \in \mathcal{O}_i(\lambda_i)$ from (d$_4$), (c$_4$) and if $S \in \mathcal{O}_i(\lambda_i)$, $S \subseteqq T_{i+1}$.

The family $\{T_1, T_2, \cdots, T_{i+1}\} (\subset \mathcal{O})$ satisfies the following conditions:

(i$_3$) $\phi = T_0 \subsetneqq T_1 \subsetneqq \cdots \subsetneqq T_i \subsetneqq T_{i+1}$, $d(T_{i+1}) < (n+1)/2$;

(ii$_3$) $\Lambda(T_0; T_1) < \Lambda(T_1; T_2) < \cdots < \Lambda(T_{i-1}; T_i) < \Lambda(T_i; T_{i+1}) < \frac{n+1-d(T_{i+1})}{2N-n+1-\#T_{i+1}}$;

(iii$_3$) When $1 \leq k \leq i+1$, for any $U \in \mathcal{O}_{k-1}$ such that $T_{k-1} \subsetneqq U$, if $d(T_{k-1}) < d(U)$, then

(a) $\Lambda(T_{k-1}; T_k) \leq \Lambda(T_{k-1}; U)$ and

(b) $\Lambda(T_{k-1}; T_k) = \Lambda(T_{k-1}; U)$ only if $U \subseteqq T_k$.

Step 3. As $d(T_{i+1}) < (n+1)/2$, we can reiterate the process given above at most $(n+1)/2$ times and then come to an end. That is to say, there exist a positive integer $p(< (n+1)/2)$ and a family $\{T_1, \cdots, T_p\}$ of subsets of X satisfying the conditions (i), (ii) and (iii) of Proposition 2.3(II). Further, when we put $\mathcal{O}_p = \{S \in \mathcal{O} \mid T_p \subset S, d(T_p) < d(S)\}$, we have the followings

(a$_5$) $\mathcal{O}_p$ is not empty.

(b$_5$) $\#\{\Lambda(T_p; S) \mid S \in \mathcal{O}_p\}$ is finite.

(c$_5$) the number $\lambda_p = \min_{S \in \mathcal{O}_p} \Lambda(T_p; S)$ satisfies the inequality

$$\lambda_p \geq (n+1-d(T_p))/(2N-n+1-\#T_p).$$

This inequality implies that (iv) of Proposition 2.3(II) holds:

(iv) For any $U \in \mathcal{O}_p$,
$\Lambda(T_p; U) \geq (n+1-d(T_p))/(2N-n+1-\#T_p)$. □

3. Generalization of Nochka weight function. Let $X, \mathcal{O}, \lambda$ etc. be as in Section 1 or 2. We give the following definition as an extension of ω and θ given in Section 1. We define a function $w : X \to (0, 1]$ and a constant h as follows:

Definition 3.1.

(I) When $\lambda \geq (n+1)/(2N-n+1)$.

For any $\boldsymbol{a} \in X$

$$w(\boldsymbol{a}) = \frac{n+1}{2N-n+1} \quad \text{and} \quad h = \frac{2N-n+1}{n+1}.$$

(II) When $\lambda < (n+1)/(2N-n+1)$.

$$w(\boldsymbol{a}) = \begin{cases} \Lambda(T_{i-1}; T_i) & \text{if } \boldsymbol{a} \in T_i - T_{i-1} \\ \dfrac{n+1-d(T_p)}{2N-n+1-\#T_p} & \text{if } \boldsymbol{a} \in X - T_p \end{cases}$$

$(i = 1, \cdots, p)$ and

$$h = (2N-n+1-\#T_p)/(n+1-d(T_p)),$$

where $T_0 = \phi$, T_i and $\Lambda(T_{i-1}; T_i)$ $(i = 1, \cdots, p)$ are those given in Proposition 2.3(II).

Like ω and θ in Theorem 1.A, the function w and the constant h have the following properties.

Theorem 3.1.

(a) *For any* $\boldsymbol{a} \in X$, $0 < hw(\boldsymbol{a}) \leq 1$;

(b) $\sum_{\boldsymbol{a} \in X}(1 - hw(\boldsymbol{a})) = 2N - n + 1 - h(n+1)$;

(c) $N/n \leq h \leq (2N-n+1)/(n+1)$;

(d) *For any* $S \in \mathcal{O}$, $\sum_{\boldsymbol{a} \in S} w(\boldsymbol{a}) \leq d(S)$.

Note 3.1. We note that

$$\{\boldsymbol{a} \in X \mid hw(\boldsymbol{a}) < 1\} = \begin{cases} \phi & \text{if } \lambda \geq (n+1)/(2N-n+1) \\ T_p & \text{if } \lambda < (n+1)/(2N-n+1). \end{cases}$$

Proof of Theorem 3.1.

(I) When $\lambda \geq (n+1)/(2N-n+1)$.

(a) For any $\boldsymbol{a} \in X$, $hw(\boldsymbol{a}) = 1$.

(b) $\sum_{\boldsymbol{a}\in X}(1-hw(\boldsymbol{a})) = 0 = 2N-n+1-h(n+1)$.

(c) $h = (2N-n+1)/(n+1)$.

(d) For any $S \in \mathcal{O}$,

$$\textstyle\sum_{\boldsymbol{a}\in S} w(\boldsymbol{a}) \leq \lambda \# S \leq (d(S)/\#S)\#S = d(S)$$

since $w(\boldsymbol{a}) = (n+1)/(2N-n+1) \leq \lambda \leq d(S)/\#S$ by the definition of λ (Definition 2.1).

(II) When $\lambda < (n+1)/(2N-n+1)$.

Let $\phi = T_0, T_1, \cdots, T_p$ be the sets obtained in Proposition 2.3(II) and let Q be a set satisfying $T_p \subset Q$ and $2N-n+1 \leq \#Q < \infty$. We choose a subset T_{p+1} of Q such that

$$T_p \subset T_{p+1} \quad \text{and} \quad \#T_{p+1} = 2N-n+1.$$

Note that $d(T_{p+1}) = n+1$.

(a) From (ii) of Proposition 2.3(II)

$$0 < hw(\boldsymbol{a}) \begin{cases} < 1 & \text{when } \boldsymbol{a} \in T_p \\ = 1 & \text{when } \boldsymbol{a} \in X - T_p. \end{cases}$$

(b) We use the following disjoint union:

$$Q = (Q - T_{p+1}) \cup (T_{p+1} - T_p) \cup \cdots \cup (T_1 - T_0).$$

$$\begin{aligned} \textstyle\sum_{\boldsymbol{a}\in Q} w(\boldsymbol{a}) &= \sum_{\boldsymbol{a}\in Q-T_{p+1}} w(\boldsymbol{a}) + \sum_{i=1}^{p+1} \sum_{\boldsymbol{a}\in T_i - T_{i-1}} w(\boldsymbol{a}) \\ &= \frac{1}{h}(\#Q - (2N-n+1)) \\ &\quad + \sum_{i=1}^{p+1}(d(T_i) - d(T_{i-1})) \\ &= \frac{1}{h}(\#Q - (2N-n+1)) + d(T_{p+1}) \\ &= \frac{1}{h}(\#Q - (2N-n+1)) + n+1, \end{aligned}$$

which implies that

$$\textstyle\sum_{\boldsymbol{a}\in Q}(1-hw(\boldsymbol{a})) = 2N-n+1-h(n+1). \tag{13}$$

Let Q be as above. Then, as $hw(\boldsymbol{a}) = 1$ for $\boldsymbol{a} \in X - Q$, we obtain the following equality from (13).

$$\begin{aligned} \textstyle\sum_{\boldsymbol{a}\in X}(1-hw(\boldsymbol{a})) &= \textstyle\sum_{\boldsymbol{a}\in Q}(1-hw(\boldsymbol{a})) \\ &= 2N-n+1-h(n+1). \end{aligned}$$

(c) As in (8) we have the inequality

$$\frac{d(T_p)}{\#T_p} < \frac{n+1}{2N-n+1} < \frac{n+1-d(T_p)}{2N-n+1-\#T_p} = h^{-1},$$

so that $h < (2N-n+1)/(n+1)$.

On the other hand, as $\#T_p - d(T_p) \leq N-n$ by Lemma 2.2 and $1 \leq d(T_p)$, we have the inequality

$$h = \frac{2N-n+1-\#T_p}{n+1-d(T_p)} \geq \frac{N+1-d(T_p)}{n+1-d(T_p)} \geq \frac{N}{n}.$$

(d) (see [3; pp.73-74]) **A**) When $d(S \cup T_p) = n+1$. By Lemma 2.1 we have the inequality

$$n+1-d(T_p) = d(S\cup T_p) - d(T_p) \leq d(S).$$

By Lemma 2.2 and (a) of this theorem we have the inequality

$$\begin{aligned} \sum_{\boldsymbol{a}\in S} w(\boldsymbol{a}) &\leq \frac{\#S}{h} \leq \frac{1}{h}(d(S) + N - n) \\ &\leq \frac{d(S)}{h}\left(1 + \frac{N-n}{d(S)}\right) \\ &\leq \frac{d(S)}{h}\left(1 + \frac{N-n}{n+1-d(T_p)}\right) \\ &= \frac{d(S)}{h}\frac{N+1-d(T_p)}{n+1-d(T_p)} \\ &= d(S)\frac{N+1-d(T_p)}{2N-n+1-\#T_p} \leq d(S) \end{aligned}$$

since $\#T_p \leq d(T_p) + N - n$ (Lemma 2.2).

B) When $d(S\cup T_p) \leq n$. Note that $\#(S \cup T_p) \leq N$. We put

$$S_i = \begin{cases} S \cap T_i & (0 \leq i \leq p) \\ S & (i = p+1). \end{cases}$$

Then, $\phi = S_0 \subset S_1 \subset \cdots \subset S_p \subset S_{p+1} = S$.

B.1) For $1 \leq i \leq p+1$, if $\#S_{i-1} < \#S_i$, then $d(T_{i-1}) < d(S_i \cup T_{i-1})$.

In fact, when $i = 1$, $0 = d(T_0) < d(S_1) = d(S_1 \cup T_0)$ as $T_0 = \phi$. When $i > 1$, suppose that

$$d(T_{i-1}) = d(S_i \cup T_{i-1}). \tag{14}$$

Then, we have that

$$d(S_i \cup T_{i-1}) - d(T_{i-2}) = d(T_{i-1}) - d(T_{i-2}) > 0$$

and that

$$\begin{aligned} \Lambda(T_{i-2}; T_{i-1}) &= \frac{d(T_{i-1}) - d(T_{i-2})}{\#T_{i-1} - \#T_{i-2}} \\ &\leq \frac{d(S_i \cup T_{i-1}) - d(T_{i-2})}{\#(S_i \cup T_{i-1}) - \#T_{i-2}} = (*) \end{aligned}$$

by (iii) of Proposition 2.3(II) since

$$T_{i-2} \subsetneqq T_{i-1} \subset S_i \cup T_{i-1},\ d(T_{i-2}) < d(S_i \cup T_{i-1})$$

and so $S_i \cup T_{i-1} \in \mathcal{O}_{i-1}$. From (14) we have

(15) $$(*) \le \frac{d(T_{i-1}) - d(T_{i-2})}{\#T_{i-1} - \#T_{i-2}} = \Lambda(T_{i-2}; T_{i-1})$$

since $\#T_{i-1} \le \#(S_i \cup T_{i-1})$. From (15) we obtain that $\#(S_i \cup T_{i-1}) = \#T_{i-1}$. Namely, $S_i \subset T_{i-1}$ and so $S_i = S_i \cap S \subset S \cap T_{i-1} = S_{i-1}$, so that $S_{i-1} = S_i$, which is a contradiction. This implies that B.1) holds.

B.2) For $i = 1, 2, \cdots, p+1$,

$$(\#S_i - \#S_{i-1})\Lambda(T_{i-1}; T_i) \le d(S_i) - d(S_{i-1}).$$

In fact, we have only to prove this inequality when

(16) $$\#S_i - \#S_{i-1} > 0.$$

Then, from B.1) $d(T_{i-1}) < d(S_i \cup T_{i-1})$. When (16) holds for $i(\le p)$, by Proposition 2.3(II)(iii)

$$\Lambda(T_{i-1}; T_i) \le \Lambda(T_{i-1}; S_i \cup T_{i-1})$$

and when (16) holds for $i = p+1$, by Proposition 2.3(II)(iv)

$$\begin{aligned}\Lambda(T_p; T_{p+1}) &= \frac{n+1-d(T_p)}{2N-n+1-\#T_p} \\ &\le \frac{d(S \cup T_p) - d(T_p)}{\#(S \cup T_p) - \#T_p} = \Lambda(T_p; S \cup T_p).\end{aligned}$$

Further, for $i = 1, 2, \cdots, p+1$, we have the relations $\#(S_i \cup T_{i-1}) = \#T_{i-1} + \#S_i - \#(S_i \cap T_{i-1})$ and by Lemma 2.1

$$d(S_i \cup T_{i-1}) \le d(T_{i-1}) + d(S_i) - d(S_i \cap T_{i-1}).$$

From these relations, we obtain that

$$\begin{aligned}\Lambda(T_{i-1}; T_i) &\le \Lambda(T_{i-1}; S_i \cup T_{i-1}) \\ &= \frac{d(S_i \cup T_{i-1}) - d(T_{i-1})}{\#(S_i \cup T_{i-1}) - \#T_{i-1}} \\ &\le \frac{d(S_i) - d(S_i \cap T_{i-1})}{\#S_i - \#(S_i \cap T_{i-1})} \\ &= \frac{d(S_i) - d(S_{i-1})}{\#S_i - \#S_{i-1}}\end{aligned}$$

since $S_i \cap T_{i-1} = (S \cap T_i) \cap T_{i-1} = S \cap T_{i-1} = S_{i-1}$. We have B.2). Now, we prove (d) when $d(S \cup T_p) \le n$. From B.2) we have the inequality

$$\begin{aligned}\sum_{\boldsymbol{a} \in S} w(\boldsymbol{a}) &\le \sum_{i=1}^{p+1} \sum_{\boldsymbol{a} \in S_i - S_{i-1}} w(\boldsymbol{a}) \\ &= \sum_{i=1}^{p+1} \Lambda(T_{i-1}; T_i)(\#S_i - \#S_{i-1}) \\ &\le d(S) - d(S_p) + \sum_{i=1}^{p} (d(S_i) - d(S_{i-1})) \\ &= d(S).\end{aligned}$$

We complete the proof of Theorem 3.1. □

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