An uncertainty principle on Sturm-Liouville hypergroups

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Abstract: As an analogue of the classical uncertainty inequality on the Euclidean space, we shall obtain a generalization on the Sturm-Liouville hypergroups $(\mathbf{R}_+, *(A))$. Especially, we shall obtain a condition on A under which the discrete part of the Plancherel formula vanishes.

Key words: uncertainty principle; Sturm-Liouville; hypergroup.

1. Sturm-Liouville hypergroups. Sturm-Liouville hypergroups are a class of one-dimensional hypergroups on $\mathbf{R}_+ = [0, \infty)$ with the convolution structure related to the second order differential operators

$$L = \frac{d^2}{dx^2} + \frac{A'(x)}{A(x)}\frac{d}{dx},$$

where A satisfies the following conditions (see [1,2]):

(1) $A > 0$ on $\mathbf{R}_+^* = (0, \infty)$, and is in $C^2(\mathbf{R}_+^*)$,

(2) on a neighborhood of 0,

$$\frac{A'(x)}{A(x)} = \frac{2\alpha+1}{x} + B(x),$$

where $\alpha \geq -\frac{1}{2}$ and (a) if $\alpha > 0$, B and B' are integrable, (b) if $\alpha = 0$, $\log xB$ and $x\log xB'$ are integrable, (c) if $-\frac{1}{2} < \alpha < 0$, $x^{2\alpha}B$ and $x^{2\alpha+1}B'$ are integrable, (d) if $\alpha = -\frac{1}{2}$, B' is integrable,

(3) $\dfrac{A'}{A} \geq 0$ on $\mathbf{R}_+^*$ and $\lim_{x\to\infty}\dfrac{A'(x)}{A(x)} = 2\rho$,

(4) $\dfrac{1}{2}\left(\dfrac{A'}{A}\right)' + \dfrac{1}{4}\left(\dfrac{A'}{A}\right)^2 - \rho^2$ is integrable at ∞.

Since $A'/A = (\log A)'$, (3) implies that A is increasing, and thus, $A(0) < \infty$. Under the conditions (1) to (3), the second order differential equation: $Lu + (\lambda^2 + \rho^2) = 0,\ \lambda \in \mathbf{C}$, has a unique solution satisfying $u(0) = 1,\ u'(0) = 0$, which we denote by ϕ_λ. Furthermore, under (4), if $\Im\lambda \geq 0$, then there exists another solution $\psi_\lambda(x)$, which behaves as $\sqrt{\pi/2}\sqrt{\lambda x}H_\alpha^{(1)}$ at ∞, where $H_\alpha^{(1)}$ is the Hankel function. Similarly, we have $\psi_\lambda^-(x)$ for $\Im\lambda \leq 0$, and for $\lambda \in \mathbf{R}_+^*$, there exists $C(\lambda) \in \mathbf{C}$ such that $\phi_\lambda(x) = C(\lambda)\psi_\lambda(x) + \overline{C(\lambda)}\psi_\lambda^-(x)$.

Let $C_{c,e}^\infty(\mathbf{R})$ denote the set of C^∞ even functions f on $\mathbf{R}$. For $f \in C_{c,e}^\infty(\mathbf{R})$ the Fourier transform $\hat{f}$ is defined by

$$\hat{f}(\lambda) = \int_0^\infty f(x)\phi_\lambda(x)A(x)dx.$$

Then the inverse transform is given as

$$f(x) = \sum_{\Lambda\in D}\pi_\Lambda\hat{f}(\Lambda)\phi_\Lambda(x) + \frac{1}{2\pi}\int_0^\infty \hat{f}(\lambda)\phi_\lambda(x)\frac{d\lambda}{|C(\lambda)|^2},$$

where D is a finite set in the interval $i(0, \rho)$ and $\pi_\lambda = \|\phi_\Lambda\|^{-2}_{L^2(\mathbf{R}_+, Adx)}$. We denote this decomposition as

$$f = {}^\circ f + f_P$$

and we call f_P and ${}^\circ f$ the principal part and the discrete part of f respectively. We denote by $\boldsymbol{F}(\nu) = (F(\lambda), \{a_\Lambda\})$ a function on $\mathbf{R}_+ \cup D$ defined by

$$\boldsymbol{F}(\nu) = \begin{cases} F(\lambda) & \text{if } \nu = \lambda \in \mathbf{R}_+ \\ a_\Lambda & \text{if } \nu = \Lambda \in D. \end{cases}$$

We put $\overline{\boldsymbol{F}}(\nu) = (\overline{F(\lambda)}, \{\overline{a_\Lambda}\})$ and define the product of $\boldsymbol{F}(\nu) = (F(\lambda), \{a_\Lambda\})$ and $\boldsymbol{G}(\nu) = (G(\lambda), \{b_\Lambda\})$ as

$$(\boldsymbol{FG})(\nu) = (F(\lambda)G(\lambda), \{a_\Lambda b_\Lambda\}).$$

Let $d\nu$ denote the measure on $\mathbf{R}_+ \cup D$ defined by

$$\int_{\mathbf{R}_+\cup D}\boldsymbol{F}(\nu)d\nu = \sum_{\Lambda\in D}\pi_\Lambda a_\Lambda + \frac{1}{2\pi}\int_0^\infty F(\lambda)|C(\lambda)|^{-2}d\lambda.$$

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For $f \in C^\infty_{c,e}(\mathbf{R})$, we put

$$\hat{\boldsymbol{f}}(\nu) = (\hat{f}(\lambda), \{\hat{f}(\Lambda)\}).$$

Then the Parseval formula on $C^\infty_{c,e}(\mathbf{R})$ can be stated as follows: For $f, g \in C^\infty_{c,e}(\mathbf{R})$

$$(5) \qquad \int_0^\infty f(x)\overline{g(x)}A(x)dx = \int_{\mathbf{R}_+\cup D} \hat{\boldsymbol{f}}(\nu)\overline{\hat{\boldsymbol{g}}(\nu)}d\nu.$$

The map $f \to \hat{\boldsymbol{f}}$, $f \in C^\infty_{c,e}(\mathbf{R})$, is extended to an isometry between $L^2(A) = L^2(\mathbf{R}_+, A(x)dx)$ and $L^2(\nu) = L^2(\mathbf{R}_+ \cup D, d\nu)$. Actually, each function f in $L^2(A)$ is of the form

$$\begin{aligned} f(x) &= \sum_{\Lambda\in D} \pi_\Lambda \hat{f}_\Lambda \phi_\Lambda(x) \\ &\quad + \frac{1}{2\pi}\int_0^\infty \hat{f}(\lambda)\phi_\lambda(x)|C(\lambda)|^{-2}d\lambda \\ &= {}^\circ f + f_P \end{aligned}$$

and their L^2-norms are given as

$$\int_0^\infty |{}^\circ f(x)|^2 A(x)dx = \sum_{\Lambda\in D} \pi_\Lambda |\hat{f}_\Lambda|^2,$$

$$\int_0^\infty |f_P(x)|^2 A(x) = \frac{1}{2\pi}\int_0^\infty |\hat{f}_P(\lambda)|^2 |C(\lambda)|^{-2}d\lambda.$$

Therefore, if we define $\hat{\boldsymbol{f}}(\nu) = (\hat{f}(\lambda), \{\hat{f}_\Lambda\})$, then $\|f\|_{L^2(A)} = \|\hat{\boldsymbol{f}}\|_{L^2(\nu)}$ holds. In particular, if $f \in C^\infty_{c,e}(\mathbf{R})$, then $\hat{f}_\Lambda = \hat{f}(\Lambda)$ for all $\Lambda \in D$.

2. Uncertainty inequality. We retain the notations in the previous sections. We put for $x \in \mathbf{R}_+$,

$$(6) \qquad a(x) = \int_0^x A(t)dt \text{ and } v(x) = \frac{a(x)}{A(x)}$$

and for $\lambda \in \mathbf{C}$,

$$w(\lambda) = (\lambda^2 + \rho^2)^{1/2}.$$

Theorem 2.1. *For all* $f \in L^1(A) \cap L^2(A)$,

$$(7) \qquad \|fv\|^2_{L^2(A)} \int_{\mathbf{R}_+\cup D} |\hat{\boldsymbol{f}}(\nu)|^2 w(\nu)^2 d\nu \geq \frac{1}{4}\|f\|^4_{L^2(A)},$$

where the equality holds if and only if f is of the form

$$f(x) = ce^{\gamma\int_0^x v(t)dt}$$

for some $c, \gamma \in \mathbf{C}$ *and* $\Re\gamma < 0$.

Proof. Without loss of generality we may suppose that $f \in C^\infty_{c,e}(\mathbf{R})$. Since $(-Lf)^\wedge(\lambda) = \hat{f}(\lambda)(\lambda^2+\rho^2) = \hat{f}(\lambda)w(\lambda)^2$ and $w(\lambda)$ is positive on $\mathbf{R}_+ \cup D$, the Parseval formula (5) yields that

$$\begin{aligned} \int_{\mathbf{R}_+\cup D} |\hat{\boldsymbol{f}}(\nu)|^2 w(\nu)^2 d\nu &= \int_0^\infty (-Lf)(x)\overline{f(x)}A(x)dx \\ &= \int_0^\infty |f'(x)|^2 A(x)dx. \end{aligned}$$

Hence it follows that

$$\begin{aligned} &\int_0^\infty |f(x)|^2 v(x)^2 A(x)dx \int_{\mathbf{R}_+\cup D} |\hat{\boldsymbol{f}}(\nu)|^2 w(\nu)^2 d\nu \\ &= \int_0^\infty |f(x)|^2 v(x)^2 A(x)dx \int_0^\infty |f'(x)|^2 A(x)dx \\ &\geq \left(\int_0^\infty \Re(f(x)f'(x))v(x)A(x)dx\right)^2 \\ &= \frac{1}{4}\left(\int_0^\infty (|f(x)|^2)' a(x)dx\right)^2 \\ &= \frac{1}{4}\left(\int_0^\infty |f(x)|^2 A(x)dx\right)^2. \end{aligned}$$

Here we used the fact that $a' = A$ (see (6)). Clearly, the equality holds if and only if $fv = cf'$ for some $c \in \mathbf{C}$, that is, $f'/f = c^{-1}v$. This means that $\log(f) = c^{-1}\int_0^x v(t)dt + C$ and thus, the desired result follows. □

Remark 2.2. When $(\mathbf{R}_+, *(A))$ is the Bessel-Kingman hypergroup, the equality holds for $e^{\gamma x^2}$, $\Re\gamma < 0$. However, when it is the Jacobi hypergroup, each function satisfying the equality has an exponential decay $e^{\gamma x}$.

Since $w^2(\lambda) = \lambda^2 + \rho^2$, (7) can be rewritten as follows:

Corollary 2.3. *Let f be the same as in Theorem* 2.1.

$$\begin{aligned} (8) \quad &\|fv\|^2_{L^2(A)} \int_{\mathbf{R}_+\cup D} |\hat{\boldsymbol{f}}(\nu)|^2 \nu^2 d\nu \\ &\geq \frac{1}{4}\|f\|^2_{L^2(A)} \int_0^\infty |f(x)|^2(1 - 4\rho^2 v(x)^2)A(x)dx. \end{aligned}$$

3. Vanishing condition of the discrete part. We shall prove that under the assumption:

$$(9) \qquad 0 \leq v(x) \leq \frac{1}{2\rho},$$

it follows that $D = \emptyset$. We suppose that $D \neq \emptyset$ and we take $f = \pi_\Lambda \phi_\Lambda$, $\Lambda \in D$. Then, since $\hat{\boldsymbol{f}}(\nu) = 1$ if $\nu = \Lambda$ and 0 otherwise, it follows from (8) that

$$\begin{aligned} &\|fv\|^2_{L^2(A)} \pi_\Lambda \Lambda^2 \\ &\geq \frac{1}{4}\|f\|^2_{L^2(A)} \int_0^\infty |f(x)|^2(1 - 4\rho^2 v(x)^2)A(x)dx. \end{aligned}$$

Here we recall that $\Lambda^2 < 0$, because $D \subset i(0, \rho)$ and $1 - 4\rho^2 v(x)^2 \geq 0$ by (9). This is contradiction. Therefore, we obtain the following

Theorem 3.1. *If* $0 \leq v \leq \dfrac{1}{2\rho}$, *then* $D = \emptyset$.

For example, if A satisfies the inequality:

$$a(x)A'(x) = \int_0^x A(x)dx \cdot A'(x) \leq A^2(x), \tag{10}$$

then A satisfies (9). Actually, (10) implies

$$v'(x) = \frac{A^2(x) - a(x)A'(x)}{A^2(x)} \geq 0.$$

Hence v is increasing on $\mathbf{R}_+$ and $v(x) = a(x)/A(x) \leq A(x)/A'(x)$ because $A/A' > 0$ by (3). Then it follows from (3) that A satisfies (9).

Corollary 3.2. *If A satisfies the inequality* (10)*, then* $D = \emptyset$.

Remark 3.3. It is well-known that $D = \emptyset$ for Chébli-Triméche hypergroups where A'/A is decreasing and (4) is not required (cf. [1]). This fact easily follows from our argument. Since A/A' is increasing and $0 \leq A/A' \leq 1/2\rho$ by (3), we see that $a \leq A/2\rho$ by integration and thus, (9) holds. Hnece $D = \emptyset$ by Theorem 3.1.

4. Uncertainty principle. We suppose that $D = \emptyset$. Then (8) is of the form:

$$\|fv\|^2_{L^2(A)} \frac{1}{2\pi} \int_0^\infty |\hat{f}(\lambda)|^2 \lambda^2 \frac{d\lambda}{|C(\lambda)|^2}$$
$$\geq \frac{1}{4} \|f\|^2_{L^2(A)} \int_0^\infty |f(x)|^2 (1 - 4\rho^2 v(x)^2) A(x)dx.$$

Since v is increasing, $v(0) = 0$, and $1 - 4\rho^2 v(x)^2 \geq 0$ by (9), it follows that f and $\hat{f}$ both cannot be concentrated around the origin.

In general, if $D \neq \emptyset$, then we must pay attention to the discrete part of f to consider uncertainty principles. We refer to [3] for the Jacobi hypergroups.

References

[1] W. R. Bloom and H. Heyer, *Harmonic analysis of probability measures on hypergroups*, de Gruyter, Berlin, 1995.

[2] O. Bracco, *Fonction maximale associée à des opérateurs de Sturm-Liouville singuliers*, Thèse, Univ. Louis Pasteur, Strasbourg, 1999.

[3] T. Kawazoe, Uncertainty principles for the Jacobi transforms, Tokyo J. Math. (to appear).