Crystals and affine Hecke algebras of type D

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Abstract: The Lascoux-Leclerc-Thibon-Ariki theory asserts that the K-group of the representations of the affine Hecke algebras of type A is isomorphic to the algebra of functions on the maximal unipotent subgroup of the group associated with a Lie algebra $\mathfrak{g}$ where $\mathfrak{g}$ is $\mathfrak{gl}_\infty$ or the affine Lie algebra $A_\ell^{(1)}$, and the irreducible representations correspond to the upper global bases. Recently, N. Enomoto and the first author presented the notion of symmetric crystals and formulated analogous conjectures for the affine Hecke algebras of type B. In this note, we present similar conjectures for certain classes of irreducible representations of affine Hecke algebras of type D. The crystal for type D is a double cover of the one for type B.

Key words: Crystal bases; affine Hecke algebras; LLT conjecture.

1. Introduction. Lascoux-Leclerc-Thibon ([3]) conjectured the relations between the representations of Hecke algebras of *type A* and the crystal bases of the affine Lie algebras of type A. Then, S. Ariki ([1]) observed that it should be understood in the setting of affine Hecke algebras and proved the LLT conjecture in a more general framework. Recently, N. Enomoto and the first author presented the notion of symmetric crystals and conjectured that certain classes of irreducible representations of the affine Hecke algebras of *type B* are described by symmetric crystals ([2]).

The purpose of this note is to formulate and explain conjectures on certain classes of irreducible representations of affine Hecke algebras of *type D* and symmetric crystals.

Let us begin by recalling the Lascoux-Leclerc-Thibon-Ariki theory. Let $\mathrm{H}_n^{\mathrm{A}}$ be the affine Hecke algebra of type A of degree n. Let $\mathrm{K}_n^{\mathrm{A}}$ be the Grothendieck group of the abelian category of finite-dimensional $\mathrm{H}_n^{\mathrm{A}}$-modules, and $\mathrm{K}^{\mathrm{A}} = \bigoplus_{n\geqslant 0}\mathrm{K}_n^{\mathrm{A}}$. Then it has a structure of Hopf algebra by the restriction and the induction functors. The set $I = \mathbf{C}^*$ may be regarded as a Dynkin diagram with I as the set of vertices and with edges between $a \in I$ and ap^2. Here p is the parameter of the affine Hecke algebra, usually denoted by q. Let $\mathfrak{g}_I$ be the associated Lie algebra, and $\mathfrak{g}_I^-$ the unipotent Lie subalgebra. Let U_I be the group associated to $\mathfrak{g}_I^-$. Hence $\mathfrak{g}_I$ is isomorphic to a direct sum of copies of $A_\ell^{(1)}$ if p^2 is a primitive ℓ-th root of unity and to a direct sum of copies of $\mathfrak{gl}_\infty$ if p has an infinite order. Then $\mathbf{C}\otimes\mathrm{K}^{\mathrm{A}}$ is isomorphic to the algebra $\mathscr{O}(U_I)$ of regular functions on U_I. Let $U_q(\mathfrak{g}_I)$ be the associated quantized enveloping algebra. Then $U_q^-(\mathfrak{g}_I)$ has an upper global basis $\{G^{\mathrm{up}}(b)\}_{b\in B(\infty)}$. By specializing $\bigoplus \mathbf{C}[q,q^{-1}]G^{\mathrm{up}}(b)$ at $q=1$, we obtain $\mathscr{O}(U_I)$. Then the LLTA theory says that the elements associated to irreducible H^{A}-modules corresponds to the image of the upper global basis.

In [2], N. Enomoto and the first author gave analogous conjectures for affine Hecke algebras of type B. In the type B case, we have to replace $U_q^-(\mathfrak{g}_I)$ and its upper global basis with a new object, the symmetric crystals. It is roughly stated as follows. Let $\mathrm{H}_n^{\mathrm{B}}$ be the affine Hecke algebra of type B of degree n. Let $\mathrm{K}_n^{\mathrm{B}}$ be the Grothendieck group of the abelian category of finite-dimensional modules over $\mathrm{H}_n^{\mathrm{B}}$, and $\mathrm{K}^{\mathrm{B}} = \oplus_{n\geqslant 0}\mathrm{K}_n^{\mathrm{B}}$. Then K^{B} has a structure of a Hopf bimodule over K^{A}. The group U_I has the anti-involution θ induced by the involution $a \mapsto a^{-1}$ of $I = \mathbf{C}^*$. Let U_I^θ be the θ-fixed point set of U_I. Then $\mathscr{O}(U_I^\theta)$ is a quotient ring of $\mathscr{O}(U_I)$. The action of $\mathscr{O}(U_I) \simeq \mathbf{C}\otimes\mathrm{K}^{\mathrm{A}}$ on $\mathbf{C}\otimes\mathrm{K}^{\mathrm{B}}$, in fact, descends to the action of $\mathscr{O}(U_I^\theta)$. They introduced the algebra $\mathcal{B}_\theta(\mathfrak{g})$, a kind of a q-analogue of the ring of differential operators on U_I^θ and then $V_\theta(\lambda)$, a q-analogue of $\mathscr{O}(U_I^\theta)$. The module $V_\theta(\lambda)$ is an irreducible $\mathcal{B}_\theta(\mathfrak{g})$-module generated by the highest weight vector ϕ_λ. Then they conjectured that:

(i) $V_\theta(\lambda)$ has a crystal basis and an upper global basis.

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(ii) K^{B} is isomorphic to a specialization of $V_\theta(\lambda)$ at $q=1$ as an $\mathscr{O}(U_I)$-module, and the irreducible representations correspond to the upper global basis of $V_\theta(\lambda)$ at $q=1$.

The representations of $\mathrm{H}_n^{\mathrm{B}}$ such that some of X_i have an eigenvalue ± 1 are excluded.

In this note, we treat the affine Hecke algebras of type D. Let $\mathrm{H}_n^{\mathrm{D}}$ be the affine Hecke algebra of type D of degree n ($\mathrm{H}_0^{\mathrm{D}}=\mathbf{C}\oplus\mathbf{C}$, $\mathrm{H}_1^{\mathrm{D}}=\mathbf{C}[X_1^{\pm}]$, see §3.1). Let $\mathrm{K}_n^{\mathrm{D}}$ be the Grothendieck group of finite-dimensional $\mathrm{H}_n^{\mathrm{D}}$-modules, and set $\mathrm{K}^{\mathrm{D}}=\bigoplus_{n\geqslant 0}\mathrm{K}_n^{\mathrm{D}}$. In D-case, we use the same algebra $\mathcal{B}_\theta(\mathfrak{g})$, but, instead of $V_\theta(\lambda)$, we use a $\mathcal{B}_\theta(\mathfrak{g})$-module V_θ generated by a pair of highest weight vectors $\phi_\pm$ (see §2.2). Our conjecture (see §3.4) is then:

(i) V_θ has a crystal basis and an upper global basis.

(ii) K^{D} is isomorphic to a specialization of V_θ at $q=1$, and the irreducible representations correspond to the upper global basis of V_θ at $q=1$.

The representations of $\mathrm{H}_n^{\mathrm{D}}$ such that some of X_i have an eigenvalue ± 1 are again excluded.

Note that the crystal basis for type D is a double cover of the one for type B.

2. Symmetric crystals.

2.1. Quantized universal enveloping algebras. We shall recall the quantized universal enveloping algebra $U_q(\mathfrak{g})$. Let I be an index set (for simple roots), and Q the free $\mathbf{Z}$-module with a basis $\{\alpha_i\}_{i\in I}$. Let $(\bullet,\bullet): Q\times Q\to\mathbf{Z}$ be a symmetric bilinear form such that $(\alpha_i,\alpha_i)/2\in\mathbf{Z}_{>0}$ for any i and $(\alpha_i^\vee,\alpha_j)\in\mathbf{Z}_{\leqslant 0}$ for $i\neq j$ where $\alpha_i^\vee:=2\alpha_i/(\alpha_i,\alpha_i)$. Let q be an indeterminate and set $\mathbf{K}:=\mathbf{Q}(q)$. We define its subrings $\mathbf{A}_0$, $\mathbf{A}_\infty$ and $\mathbf{A}$ as follows:

$$\mathbf{A}_0=\{f/g;\, f(q),g(q)\in\mathbf{Q}[q],\, g(0)\neq 0\},$$
$$\mathbf{A}_\infty=\{f/g;\, f(q^{-1}),g(q^{-1})\in\mathbf{Q}[q^{-1}],\, g(0)\neq 0\},$$
$$\mathbf{A}=\mathbf{Q}[q,q^{-1}].$$

Definition 2.1. *The quantized universal enveloping algebra $U_q(\mathfrak{g})$ is the $\mathbf{K}$-algebra generated by the elements e_i, f_i and invertible elements t_i ($i\in I$) with the following defining relations.*

(1) *The t_i's commute with each other.*

(2) $t_je_it_j^{-1}=q^{(\alpha_j,\alpha_i)}e_i$ *and* $t_jf_it_j^{-1}=q^{-(\alpha_j,\alpha_i)}f_i$ *for any* $i,j\in I$.

(3) $[e_i,f_j]=\delta_{ij}\dfrac{t_i-t_i^{-1}}{q_i-q_i^{-1}}$ *for* $i,\ j\in I$, *where* $q_i:=q^{(\alpha_i,\alpha_i)/2}$.

(4) (Serre relation) *For* $i\neq j$,

$$\sum_{k=0}^{b}(-1)^k e_i^{(k)}e_je_i^{(b-k)}=\sum_{k=0}^{b}(-1)^k f_i^{(k)}f_jf_i^{(b-k)}=0.$$

Here $b=1-(\alpha_i^\vee,\alpha_j)$ *and*

$$e_i^{(k)}=e_i^k/[k]_i!,\ f_i^{(k)}=f_i^k/[k]_i!,$$
$$[k]_i=(q_i^k-q_i^{-k})/(q_i-q_i^{-1}),\ [k]_i!=[1]_i\cdots[k]_i.$$

Let us denote by $U_q^-(\mathfrak{g})$ (resp. $U_q^+(\mathfrak{g})$) the subalgebra of $U_q(\mathfrak{g})$ generated by the f_i's (resp. the e_i's). Let e_i' and e_i^* be the operators on $U_q^-(\mathfrak{g})$ defined by

$$[e_i,a]=\frac{(e_i^*a)t_i-t_i^{-1}e_i'a}{q_i-q_i^{-1}}\quad (a\in U_q^-(\mathfrak{g})).$$

Then these operators satisfy the following formula similar to derivations:

$$e_i'(ab)=e_i'(a)b+(\mathrm{Ad}(t_i)a)e_i'b,$$
$$e_i^*(ab)=ae_i^*b+(e_i^*a)(\mathrm{Ad}(t_i)b).$$

The algebra $U_q^-(\mathfrak{g})$ has a unique symmetric bilinear form $(\bullet,\bullet)$ such that $(1,1)=1$ and

$$(e_i'a,b)=(a,f_ib)\quad\text{for any } a,b\in U_q^-(\mathfrak{g}).$$

It is non-degenerate and satisfies $(e_i^*a,b)=(a,bf_i)$.

2.2. Symmetry. Let θ be an automorphism of I such that $\theta^2=\mathrm{id}$ and $(\alpha_{\theta(i)},\alpha_{\theta(j)})=(\alpha_i,\alpha_j)$. Hence it extends to an automorphism of the root lattice Q by $\theta(\alpha_i)=\alpha_{\theta(i)}$, and induces an automorphism of $U_q(\mathfrak{g})$.

Let $\mathcal{B}_\theta(\mathfrak{g})$ be the $\mathbf{K}$-algebra generated by E_i, F_i, and invertible elements K_i ($i\in I$) satisfying the following defining relations:

$$(2.1)\quad\begin{cases}\text{(i)} & \text{the } K_i\text{'s commute with each other,}\\ \text{(ii)} & K_{\theta(i)}=K_i \text{ for any } i\in I,\\ \text{(iii)} & K_iE_jK_i^{-1}=q^{(\alpha_i+\alpha_{\theta(i)},\alpha_j)}E_j\\ & \text{and } K_iF_jK_i^{-1}=q^{(\alpha_i+\alpha_{\theta(i)},-\alpha_j)}F_j\\ & \text{for } i,j\in I,\\ \text{(iv)} & E_iF_j=q^{-(\alpha_i,\alpha_j)}F_jE_i+(\delta_{i,j}+\delta_{\theta(i),j}K_i)\\ & \text{for } i,j\in I,\\ \text{(v)} & \text{the } E_i\text{'s and the } F_i\text{'s satisfy the Serre}\\ & \text{relations.}\end{cases}$$

Hence $\mathcal{B}_\theta(\mathfrak{g})\simeq U_q^-(\mathfrak{g})\otimes\mathbf{K}[K_i^{\pm1}; i\in I]\otimes U_q^+(\mathfrak{g})$. We set $E_i^{(n)}=E_i^n/[n]_i!$ and $F_i^{(n)}=F_i^n/[n]_i!$.

Proposition 2.2.

(i) *There exists a $\mathcal{B}_\theta(\mathfrak{g})$-module V_θ generated by linearly independent vectors ϕ_+ and ϕ_- such that*

(a) $E_i\phi_\pm=0$ *for any* $i\in I$,

(b) $K_i\phi_\pm = \phi_\mp$ *for any* $i \in I$,

(c) $\{u \in V_\theta; E_i u = 0$ *for any* $i \in I\}$
$= \mathbf{K}\phi_+ \oplus \mathbf{K}\phi_-$.

Moreover such a V_θ *is unique up to an isomorphism.*

(ii) *There exists a unique symmetric bilinear form* $(\bullet,\bullet)$ *on* V_θ *such that* $(\phi_{\varepsilon_1}, \phi_{\varepsilon_2}) = \delta_{\varepsilon_1,\varepsilon_2}$ *for* $\varepsilon_1, \varepsilon_2 \in \{+,-\}$ *and* $(E_i u, v) = (u, F_i v)$ *for any* $i \in I$ *and* $u, v \in V_\theta$, *and it is non-degenerate.*

Such a V_θ is constructed as follows. Let $\mathscr{S}$ be the quantum shuffle algebra (see [5]) generated by words $\langle i_1, \dots, i_l\rangle$ for $i_1, \dots, i_l \in I$ and $l \geq 1$ and ϕ''_+ and ϕ''_- as two empty words. We assign to a word $\langle i_1, \dots, i_l\rangle$ the weight $-(\alpha_{i_1} + \cdots + \alpha_{i_l})$. We define the actions of E_i, F_i and K_i on $\mathscr{S}$ as follows:

$$
\begin{aligned}
&F_i\phi''_+ = \langle i\rangle, \quad F_i\phi''_- = \langle \theta i\rangle,\\
&E_i\langle j\rangle = \delta_{i,j}\phi''_+ + \delta_{i,\theta j}\phi''_-,\\
&K_i\phi''_\pm = \phi''_\mp,\\
&K_i\langle i_1, \dots, i_l\rangle = q^{-(\alpha_i + \alpha_{\theta(i)}, \alpha_{i_1} + \cdots + \alpha_{i_l})}\\
&\qquad \cdot \langle i_1, \dots, i_{l-1}, \theta(i_l)\rangle,\\
&E_i\langle i_1, \dots, i_l\rangle = \delta_{i,i_1}\langle i_2, \dots, i_l\rangle,\\
&F_i\langle i_1, \dots, i_l\rangle = \langle i\rangle * \langle i_1, \dots, i_l\rangle\\
&\qquad + q^{(\alpha_i, \mathrm{wt}(\langle i_1, \dots, i_{l-1}, \theta(i_l)\rangle))}\langle i_1, \dots, i_{l-1}, \theta(i_l)\rangle * \langle \theta i\rangle\\
&\quad = \sum_{\nu=0}^{l} q^{-(\alpha_i, \alpha_{i_1} + \cdots + \alpha_{i_\nu})}\langle i_1, \dots, i_\nu, i, i_{\nu+1}, \dots, i_l\rangle\\
&\qquad + q^{-(\alpha_i, \alpha_1 + \cdots + \alpha_{i_{l-1}} + \alpha_{\theta(i_l)})}\\
&\sum_{\nu=0}^{l} q^{-(\alpha_{\theta(i)}, \alpha_{\nu+1} + \cdots + \alpha_{i_{l-1}} + \alpha_{\theta(i_l)})}\\
&\qquad \cdot \langle i_1, \dots, i_\nu, \theta(i), i_{\nu+1}, \dots, i_{l-1}, \theta(i_l)\rangle
\end{aligned}
$$

for $i, j \in I$, $l \geqslant 1$ and $i_1, \dots, i_l \in I$.

Then the operators E_i, F_i and K_i satisfy the commutation relations (2.1) except the Serre relations for the E_i's.

Consider the $U_q^-(\mathfrak{g})$-module $V' = U_q^-(\mathfrak{g})\phi'_+ \oplus U_q^-(\mathfrak{g})\phi'_-$ generated by a pair of vacuum vectors $\phi'_\pm$. There exists a unique $U_q^-(\mathfrak{g})$-linear map $\psi : V' \to \mathscr{S}$ such that $\phi'_\pm \mapsto \phi''_\pm$. We define an action of $\mathcal{B}_\theta(\mathfrak{g})$ on V' by

$$
\begin{aligned}
K_i(a\phi'_\pm) &= (\mathrm{Ad}(t_i t_{\theta(i)})a)\phi'_\mp,\\
E_i(a\phi'_\pm) &= e'_i(a)\phi'_\pm + Ad(t_i)(e^*_{\theta i}(a))\phi'_\mp,\\
F_i(a\phi'_\pm) &= f_i a\phi'_\pm
\end{aligned}
$$

for $a \in U_q^-(\mathfrak{g})$. Then ψ commutes with the actions of E_i, F_i and K_i, and its image $\psi(V')$ is V_θ.

Hereafter we assume further that

(2.2) there is no $i \in I$ such that $\theta(i) = i$.

Under this condition, we conjecture that V_θ has a crystal basis. This means the following. We define the modified root operators:

$$\widetilde{E}_i(u) = \sum_{n\geqslant 1} F_i^{(n-1)}u_n \text{ and } \widetilde{F}_i(u) = \sum_{n\geqslant 0} F_i^{(n+1)}u_n$$

when writing $u = \sum_{n\geqslant 0} F_i^{(n)}u_n$ with $E_i u_n = 0$. Let L_θ be the $\mathbf{A}_0$-submodule of V_θ generated by $\widetilde{F}_{i_1}\cdots\widetilde{F}_{i_\ell}\phi_\pm$ ($\ell \geqslant 0$ and $i_1, \dots, i_\ell \in I$), and define the subset $B_\theta \subset L_\theta/qL_\theta$ by:

$$B_\theta := \{\widetilde{F}_{i_1}\cdots\widetilde{F}_{i_\ell}\phi_\pm \bmod qL_\theta; \ell \geqslant 0, i_1, \dots, i_\ell \in I\}.$$

Conjecture 2.3.

(i) $\widetilde{F}_i L_\theta \subset L_\theta$ and $\widetilde{E}_i L_\theta \subset L_\theta$,

(ii) B_θ is a basis of L_θ/qL_θ,

(iii) $\widetilde{F}_i B_\theta \subset B_\theta$, and $\widetilde{E}_i B_\theta \subset B_\theta \sqcup \{0\}$.

Moreover we conjecture that V_θ has a global crystal basis. Namely, let $-$ be the bar-operator of V_θ, which is characterized by: $\overline{q} = q^{-1}$, $-$ commutes with the E_i's, and $(\phi_\pm)^- = \phi_\pm$ (such an operator exists). Let us denote by $\mathcal{B}_\theta(\mathfrak{g})^{\mathrm{up}}_{\mathbf{A}}$ the $\mathbf{A}$-subalgebra of $\mathcal{B}_\theta(\mathfrak{g})$ generated by $E_i^{(n)}$, F_i and $K_i^{\pm 1}$ ($i \in I$). Let $(V_\theta)_{\mathbf{A}}$ be the largest $\mathcal{B}_\theta(\mathfrak{g})^{\mathrm{up}}_{\mathbf{A}}$-submodule of V_θ such that $(V_\theta)_{\mathbf{A}} \cap (\mathbf{K}\phi_+ + \mathbf{K}\phi_-) = \mathbf{A}\phi_+ + \mathbf{A}\phi_-$.

Conjecture 2.4. $(L_\theta, L_\theta^-, (V_\theta)_{\mathbf{A}})$ is balanced.

Namely, $E := L_\theta \cap L_\theta^- \cap (V_\theta)_{\mathbf{A}} \to L_\theta/qL_\theta$ is an isomorphism. Let $G^{\mathrm{up}}\colon L_\theta/qL_\theta \xrightarrow{\sim} E$ be its inverse. Then $\{G^{\mathrm{up}}(b); b \in B_\theta\}$ forms a basis of V_θ. We call this basis the *upper global basis* of V_θ.

Remark 2.5. Assume that Conjectures 2.3 and 2.3 hold.

(i) We have $\{b \in B_\theta; \widetilde{E}_i b = 0 \text{ for any } i \in I\} = \{\phi_+, \phi_-\}$.

(ii) There exists a unique involution σ of the $\mathcal{B}_\theta(\mathfrak{g})$-module V_θ such that $\sigma(\phi_\pm) = \phi_\mp$. It extends to the involution σ of $\mathscr{S}$ by $\sigma(\langle i_1, \dots, i_l\rangle) = \langle i_1, \dots, i_{l-1}, \theta(i_l)\rangle$. It induces also involutions of L_θ and B_θ.

(iii) We have $\sigma(b) \neq b$ for any $b \in B_\theta$.

(iv) We conjecture that $\widetilde{F}_i b \neq \widetilde{F}_j b$ for any $b \in B_\theta$ and $i \neq j \in I$.

(v) In [2], a $\mathcal{B}_\theta(\mathfrak{g})$-module $V_\theta(\lambda) = \mathcal{B}_\theta(\mathfrak{g})\phi_\lambda$ and its crystal basis $B_\theta(\lambda)$ are introduced. We have a monomorphism of $\mathcal{B}_\theta(\mathfrak{g})$-modules

$$\iota : V_\theta(\lambda) \rightarrowtail V_\theta$$

with $\lambda = 0$, which sends ϕ_λ to $\phi_+ + \phi_-$. Its image coincides with $\{v \in V_\theta; \sigma(v) = v\}$.

Any element $b \in B_\theta(\lambda)$ is sent to $b' + \sigma(b')$ for some $b' \in B_\theta$. Moreover, we have $\iota(G^{\mathrm{up}}(b)) = G^{\mathrm{up}}(b') + \sigma(G^{\mathrm{up}}(b'))$. In particular, we have

$$B_\theta(\lambda) \simeq B_\theta/\sim.$$

Here $\sim$ is the equivalence relation given by $b \sim \sigma b$.

3. Affine Hecke algebra of type D.

3.1. Definition. For $p \in \mathbf{C}^*$ and $n \in \mathbf{Z}_{\geqslant 2}$, the affine Hecke algebra $\mathrm{H}_n^{\mathrm{D}}$ of type D_n is the $\mathbf{C}$-algebra generated by T_i $(0 \leqslant i < n)$ and invertible elements X_i $(1 \leqslant i \leqslant n)$ satisfying the defining relations:

(i) the X_i's commute with each other,

(ii) the T_i's satisfy the braid relation: $T_1T_0 = T_0T_1$, $T_0T_2T_0 = T_2T_0T_2$, $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$ $(1 \leqslant i < n-1)$, $T_iT_j = T_jT_i$ $(1 \leqslant i < j-1 < n-1$ or $i = 0 < 3 \leqslant j < n)$,

(iii) $(T_i - p)(T_i + p^{-1}) = 0$ $(0 \leqslant i < n)$,

(iv) $T_0X_1^{-1}T_0 = X_2$, $T_iX_iT_i = X_{i+1}$ $(1 \leqslant i < n)$, and $T_iX_j = X_jT_i$ if $1 \leq i \neq j, j-1$ or $i = 0$ and $j \geq 3$.

We define $\mathrm{H}_0^{\mathrm{D}} = \mathbf{C} \oplus \mathbf{C}$ and $\mathrm{H}_1^{\mathrm{D}} = \mathbf{C}[X_1^{\pm 1}]$.

We assume that $p \in \mathbf{C}^*$ satisfies

$$p^2 \neq 1. \tag{3.1}$$

Let us denote by $\mathbf{Pol}_n$ the Laurent polynomial ring $\mathbf{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$, and by $\widetilde{\mathbf{Pol}}_n$ its quotient field $\mathbf{C}(X_1, \ldots, X_n)$. Then $\mathrm{H}_n^{\mathrm{D}}$ is isomorphic to the tensor product of $\mathbf{Pol}_n$ and the subalgebra generated by the T_i's that is isomorphic to the Hecke algebra of type D_n. We have

$$T_ia = (s_ia)T_i + (p - p^{-1})\frac{a - s_ia}{1 - X^{-\alpha_i^\vee}} \quad \text{for } a \in \mathbf{Pol}_n.$$

Here, $X^{-\alpha_i^\vee} = X_1^{-1}X_2^{-1}$ $(i = 0)$ and $X^{-\alpha_i^\vee} = X_iX_{i+1}^{-1}$ $(1 \leqslant i < n)$. The s_i's are the Weyl group action on $\mathbf{Pol}_n$: $(s_0a)(X_1, \ldots, X_n) = a(X_2^{-1}, X_1^{-1}, \ldots, X_n)$ and $(s_ia)(X_1, \ldots, X_n) = a(X_1, \ldots, X_{i+1}, X_i, \ldots, X_n)$ for $1 \leqslant i < n$.

3.2. Intertwiner. The algebra $\mathrm{H}_n^{\mathrm{D}}$ acts faithfully on $\mathrm{H}_n^{\mathrm{D}}/\sum_i \mathrm{H}_n^{\mathrm{D}}(T_i - p) \simeq \mathbf{Pol}_n$. Set $\varphi_i = (1 - X^{-\alpha_i^\vee})T_i - (p - p^{-1}) \in \mathrm{H}_n^{\mathrm{D}}$ and $\tilde{\varphi}_i = (p^{-1} - pX^{-\alpha_i^\vee})^{-1}\varphi_i \in \widetilde{\mathbf{Pol}}_n \otimes_{\mathbf{Pol}_n} \mathrm{H}_n^{\mathrm{D}}$. Then the action of $\tilde{\varphi}_i$ on $\mathbf{Pol}_n$ coincides with s_i. They are called intertwiners.

3.3. Affine Hecke algebra of type A. The affine Hecke algebra $\mathrm{H}_n^{\mathrm{A}}$ of type A_n is isomorphic to the subalgebra of $\mathrm{H}_n^{\mathrm{D}}$ generated by T_i $(1 \leqslant i < n)$ and $X_i^{\pm 1}$ $(1 \leqslant i \leqslant n)$. For a finite-dimensional $\mathrm{H}_n^{\mathrm{A}}$-module M, let us decompose

$$M = \bigoplus_{a \in (\mathbf{C}^*)^n} M_a \tag{3.2}$$

where

$$M_a = \{u \in M; (X_i - a_i)^N u = 0 \text{ for any } i \text{ and } N \gg 0\}$$

for $a = (a_1, \ldots, a_n) \in (\mathbf{C}^*)^n$. For a subset $I \subset \mathbf{C}^*$, we say that M is of type I if all the eigenvalues of X_i belong to I. The group $\mathbf{Z}$ acts on $\mathbf{C}^*$ by $\mathbf{Z} \ni n : a \mapsto ap^{2n}$. By well-known results in type A, it is enough to treat the irreducible modules of type I for an orbit I with respect to the $\mathbf{Z}$-action on $\mathbf{C}^*$ in order to study the irreducible modules over the affine Hecke algebras of type A.

3.4. Representations of affine Hecke lgebras of type D. For $n, m \geqslant 0$, set $\mathbf{F}_{n,m} := \mathbf{C}[X_1^{\pm 1}, \ldots, X_{n+m}^{\pm 1}, D^{-1}]$ where

$$\begin{aligned} D := \prod_{1 \leqslant i \leqslant n < j \leqslant n+m} & (X_i - p^2X_j)(X_i - p^{-2}X_j) \\ & \cdot (X_i - p^2X_j^{-1})(X_i - p^{-2}X_j^{-1}) \\ & \cdot (X_i - X_j)(X_i - X_j^{-1}). \end{aligned}$$

Then we can embed $\mathrm{H}_m^{\mathrm{D}}$ into $\mathrm{H}_{n+m}^{\mathrm{D}} \otimes_{\mathbf{Pol}_{n+m}} \mathbf{F}_{n,m}$ by

$$\begin{aligned} & T_0 \mapsto \tilde{\varphi}_n \cdots \tilde{\varphi}_1\tilde{\varphi}_{n+1} \cdots \tilde{\varphi}_2T_0\tilde{\varphi}_2 \cdots \tilde{\varphi}_{n+1}\tilde{\varphi}_1 \cdots \tilde{\varphi}_n, \\ & T_i \mapsto T_{i+n} \quad (1 \leqslant i < m), \\ & X_i \mapsto X_{i+n} \quad (1 \leqslant i \leqslant m). \end{aligned}$$

Its image commutes with $\mathrm{H}_n^{\mathrm{D}} \subset \mathrm{H}_{n+m}^{\mathrm{D}}$. Hence $\mathrm{H}_{n+m}^{\mathrm{D}} \otimes_{\mathbf{Pol}_{n+m}} \mathbf{F}_{n,m}$ is a right $\mathrm{H}_n^{\mathrm{D}} \otimes \mathrm{H}_m^{\mathrm{D}}$-module.

For a finite-dimensional $\mathrm{H}_n^{\mathrm{D}}$-module M, we decompose M as in (3.2). The semidirect product group $\mathbf{Z}_2 \ltimes \mathbf{Z} = \{1, -1\} \ltimes \mathbf{Z}$ acts on $\mathbf{C}^*$ by $(\epsilon, n) : a \mapsto a^\epsilon p^{2n}$.

Let I and J be $\mathbf{Z}_2 \ltimes \mathbf{Z}$-invariant subsets of $\mathbf{C}^*$ such that $I \cap J = \emptyset$. Then for an $\mathrm{H}_n^{\mathrm{D}}$-module N of type I and $\mathrm{H}_m^{\mathrm{D}}$-module M of type J, the action of $\mathbf{Pol}_{n+m}$ on $N \otimes M$ extends to an action of $\mathbf{F}_{n,m}$. We set

$$N \diamond M := (\mathrm{H}_{n+m}^{\mathrm{D}} \otimes_{\mathbf{Pol}_{n+m}} \mathbf{F}_{n,m}) \underset{(\mathrm{H}_n^{\mathrm{D}} \otimes \mathrm{H}_m^{\mathrm{D}}) \otimes_{\mathbf{Pol}_{n+m}} \mathbf{F}_{n,m}}{\otimes} (N \otimes M).$$

Lemma 3.1.

(i) *Let N be an irreducible $\mathrm{H}_n^{\mathrm{D}}$-module of type I and M an irreducible $\mathrm{H}_m^{\mathrm{D}}$-module of type J. Then $N \diamond M$ is an irreducible $\mathrm{H}_{n+m}^{\mathrm{D}}$-module of type $I \cup J$.*

(ii) *Conversely if L is an irreducible $\mathrm{H}_n^{\mathrm{D}}$-module of type $I \cup J$, then there exists an integer m $(0 \leqslant m \leqslant n)$, an irreducible $\mathrm{H}_m^{\mathrm{D}}$-module N of* *type I and an irreducible $\mathrm{H}^{\mathrm{D}}_{n-m}$-module M of type J such that $L \simeq N \diamond M$.*

(iii) *Assume that a $\mathbf{Z}_2 \times \mathbf{Z}$-orbit I decomposes into $I = I_+ \sqcup I_-$ where $I_\pm$ are $\mathbf{Z}$-orbits and $I_- = (I_+)^{-1}$. Then for any irreducible $\mathrm{H}^{\mathrm{D}}_n$-module L of type I, there exists an irreducible $\mathrm{H}^{\mathrm{A}}_n$-module M such that $L \simeq \mathrm{Ind}^{\mathrm{H}^{\mathrm{D}}_n}_{\mathrm{H}^{\mathrm{A}}_n} M$.*

Hence in order to study H^{D}-modules, it is enough to study irreducible modules of type I for a $\mathbf{Z}_2 \times \mathbf{Z}$-orbit I in $\mathbf{C}^*$ such that I is a $\mathbf{Z}$-orbit, namely $I = \pm\{p^n; n \in \mathbf{Z}_{\mathrm{odd}}\}$ or $I = \pm\{p^n; n \in \mathbf{Z}_{\mathrm{even}}\}$.

For a $\mathbf{Z}_2 \times \mathbf{Z}$-invariant subset I of $\mathbf{C}^*$, we define $\mathrm{K}^{\mathrm{D}}_{I,n}$ to be the Grothendieck group of the abelian category of finite-dimensional $\mathrm{H}^{\mathrm{D}}_n$-modules of type I. We set $\mathrm{K}^{\mathrm{D}}_I = \bigoplus_{n \geqslant 0} \mathrm{K}^{\mathrm{D}}_{I,n}$.

We take the case

$$I = \{p^n; n \in \mathbf{Z}_{\mathrm{odd}}\}$$

and assume that any of ± 1 is not contained in I. The set I may be regarded as the set of vertices of a Dynkin diagram. Let us define an automorphism θ of I by $a \mapsto a^{-1}$. Let $\mathfrak{g}_I$ be the associated Lie algebra ($\mathfrak{g}_I$ is isomorphic to $\mathfrak{gl}_\infty$ if p has an infinite order, and isomorphic to $A^{(1)}_\ell$ if p^2 is a primitive ℓ-th root of unity).

For a finite-dimensional $\mathrm{H}^{\mathrm{D}}_n$-module M and $a \in I$, let $E_a M$ be the generalized a-eigenspace of X_n on M, regarded as an $\mathrm{H}^{\mathrm{D}}_{n-1}$-module. Let $F_a M$ be the $\mathrm{H}^{\mathrm{D}}_{n+1}$-module $\mathrm{Ind}^{\mathrm{H}^{\mathrm{D}}_{n+1}}_{\mathrm{H}^{\mathrm{D}}_n \otimes \mathbf{C}[X^{\pm}_{n+1}]}(M \otimes (a))$ where (a) is the 1-dimensional representation of $\mathbf{C}[X^{\pm 1}_{n+1}]$ on which X_{n+1} acts as a. Then E_a and F_a are exact functors and define $E_a : \mathrm{K}^{\mathrm{D}}_{I,n} \to \mathrm{K}^{\mathrm{D}}_{I,n-1}$ and $F_a : \mathrm{K}^{\mathrm{D}}_{I,n} \to \mathrm{K}^{\mathrm{D}}_{I,n+1}$.

For an irreducible $M \in \mathrm{K}^{\mathrm{D}}_{I,n}$ and $a \in I$, define $\tilde{e}_a M \in \mathrm{K}^{\mathrm{D}}_{I,n-1}$ to be the socle of $E_a M$. Define $\tilde{f}_a M \in \mathrm{K}^{\mathrm{D}}_{I,n+1}$ to be the cosocle of $F_a M$. In fact, $\tilde{f}_a M$ is always irreducible, and $\tilde{e}_a M$ is a zero module or irreducible.

The ring $\mathrm{H}^{\mathrm{D}}_0 = \mathbf{C} \oplus \mathbf{C}$ has two irreducible modules $\phi_\pm$. We understand

$$E_a((b)) = \tilde{e}_a((b)) = \begin{cases} \phi_\pm & \text{if } a = b^{\pm 1}, \\ 0 & \text{otherwise}, \end{cases}$$

$$F_a(\phi_\pm) = \tilde{f}_a(\phi_\pm) = (a^{\pm 1}).$$

Let V_θ be as in Proposition 2.2.

Conjecture 3.2.

(i) K^{D} is isomorphic to $(V_\theta)_{\mathbf{A}}/(q-1)(V_\theta)_{\mathbf{A}}$.

(ii) V_θ has a crystal basis and an upper global basis.

(iii) The elements of $\mathrm{K}^{\mathrm{D}}_I$ associated to irreducible representations correspond to the upper global basis of V_θ at $q = 1$.

(iv) The operators $\tilde{F}_i$ and $\tilde{E}_i$ correspond to $\tilde{f}_i$ and $\tilde{e}_i$, respectively.

Consider $\tilde{H} = \mathrm{H}^{\mathrm{D}}_n \otimes \mathbf{C}[\theta]/(\theta^2 - 1)$ with multiplication $\theta T_1 = T_0 \theta$, $X_1 \theta = \theta X_1^{-1}$ and θ commuting with all other generators. Then $\tilde{H}$ is isomorphic to the specialization of the affine Hecke algebra of type B in which the generator for the node corresponding to the short root has eigenvalues ± 1. This explains why the crystal graph in the above case is a double covering of the crystal graph for the same $\mathbf{Z}_2 \times \mathbf{Z}$-orbit in type B. (See Remark 2.5 (v).)

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