Strong unique continuation property for some second order elliptic systems

By Makoto SANADA

Department of Mathematics, Osaka University, Machikaneyama 1-16, Toyonaka, Osaka 560-0043, Japan

(Communicated by Heisuke HIRONAKA, M.J.A., Sept. 12, 2007)

Abstract: We study the unique continuation property of some second order elliptic systems.

Key words: Strong unique continuation; elliptic system.

1. Introduction. In this paper we prove the strong unique continuation property for some second order systems. There are many results for second order single equation (for example [1–3]). Let Ω be a nonempty open connected subset of $\mathbf{R}^n$ containing the origin, and let

$$p(x,\partial) = \sum_{j,k} a_{j,k}(x)\partial_j\partial_k$$

be an elliptic differential operator in Ω such that $a_{j,k}(0)$ is real and $a_{j,k}(x)$ is Lipschitz continuous in Ω. In [3], Regbaoui proved that if $u \in H^1_{loc}(\Omega)$ satisfies

$$|p(x,\partial)u| \le C_0|x|^{-2}|u| + C_1|x|^{-1}|\nabla u|$$

with a sufficiently small C_1 and

$$\lim_{\rho\to 0}\rho^{-\beta}\int_{|x|\le\rho}|u|^2dx = 0$$

for any positive β, then u is identically zero in Ω.

We are interested in second order systems, that is, $a_{j,k}(x)$ is of matrix valued.

2. Main Results. Let Ω be a nonempty open connected subset of $\mathbf{R}^n$ containing the origin, and let

$$P(x,\partial) = \sum_{1\le j,k\le n} A_{j,k}(x)\partial_j\partial_k \tag{2.1}$$

be an elliptic differential operator in Ω where $A_{j,k}$ is an $N\times N$ matrix valued function with the entries which are Lipschitz continuous in Ω for any $1 \le j,k \le n$. We assume that $P(x,\partial)$ satisfies the following properties;

$$A^*_{j,k}A_{l,m} = A_{l,m}A^*_{j,k} \tag{2.2}$$

for any $1 \le j,k,l,m \le n$, and there exist an elliptic differential operator $p(\partial) = \sum_{1\le j,k\le n} a_{j,k}\partial_j\partial_k$ with real coefficients and complex numbers λ_j $j = 1,2\dots,N$ such that

$$P(0,\partial) = \operatorname{diag}\begin{pmatrix}\lambda_1 p(\partial) & & \\ & \ddots & \\ & & \lambda_N p(\partial)\end{pmatrix}. \tag{2.3}$$

Then it follows the following theorem.

Theorem 2.1. *There exists a positive constant C^* depending only on $p(\partial)$ such that if $u \in \{H^2_{loc}(\Omega)\}^N$ satisfies*

$$|P(x,\partial)u| \le C_0|u|/|x|^2 + C_1|\nabla u|/|x| \tag{2.4}$$

with $C_1 < C^*$ and

$$\lim_{\rho\to 0}\rho^{-\beta}\int_{|x|\le\rho}|\partial^\alpha_x u|^2dx = 0 \tag{2.5}$$

for any positive β and any $|\alpha| \le 2$, then u is identically zero in some neighborhood of the origin.

3. Proof of Theorem. After a linear transform, we may assume that $p(\partial) = \triangle$. Considering $\tilde{u} = (\lambda_1^{-1}u_1,\dots,\lambda_N^{-1}u_N)$, without loss of generality, it suffices to prove the theorem assuming $P(0,\partial) = \triangle I_N$. In [3] Regbaoui proved the following result.

Lemma 3.1. *There exists a positive constant C such that*

$$\sum_{|\alpha|\le 2}\beta^{2-2|\alpha|}\int|x|^{-2\beta+2|\alpha|-3}|\partial^\alpha_x u|^2dx \le C\int|x|^{-2\beta+1}|\triangle u|^2dx \tag{3.1}$$

2000 Mathematics Subject Classification. Primary 35B60.

for any $\beta \in \{j+1/2; j \in \mathbf{N}\}$ *and any* $u \in C_0^\infty(\Omega \setminus \{0\})$.

Remark 3.2. The estimate (3.1) in Lemma remains valid if we assume $u \in \{H^2_{loc}(\Omega)\}^N$ with compact support satisfies (2.5).

Proposition 3.3. *Let* $u \in \{H^2_{loc}(\Omega)\}^N$ *satisfy* (2.4) *and* (2.5). *Then there exist positive* C_2 *and* C_3 *such that*

$$\|u\|^2_{H^2(\{x;|x|\le\rho\})} \le C_2 \exp(-C_3\rho^{-1})$$

for any small positive ρ.

Proof. Let $\chi(r) \in C_0^\infty([0,\infty))$ be a nonnegative function such that $\chi(r) = 1$ when $0 \le r \le 1$, $\chi(r) = 0$ when $2 \le r$ and $|\chi'| \le C$. Setting $\tilde{u}(x) = \chi(\epsilon^{-1}\beta|x|)u(x)$ where ϵ is a small positive parameter which will be determined later. By Lemma 3.1 and (2.3) we have

$$\sum_{|\alpha|\le 2} \beta^{2-2|\alpha|} \int |x|^{-2\beta+2|\alpha|-3}|\partial_x^\alpha \tilde{u}|^2 dx \le C \int |x|^{-2\beta+1}|\triangle \tilde{u}|^2 dx \le C \int |x|^{-2\beta+1}|P(0,\partial)\tilde{u}|^2 dx. \tag{3.2}$$

Since $A_{j,k}$ is Lipschitz continuous and $|x| \le 2\epsilon^{-1}\beta$, it follows that

$$\int |x|^{-2\beta+1}|P(0,\partial)\tilde{u}|^2 dx \le 2\int |x|^{-2\beta+1}|P(x,\partial)\tilde{u}|^2 dx + 2\sum_{|\alpha|\le 2}\int |x|^{-2\beta+3}|\partial_x^\alpha \tilde{u}|^2 dx \tag{3.3}$$

and

$$\sum_{|\alpha|\le 2}\int |x|^{-2\beta+3}|\partial_x^\alpha \tilde{u}|^2 dx \le 4\epsilon^2\beta^{-2}\sum_{|\alpha|\le 2}\int |x|^{-2\beta+1}|\partial_x^\alpha \tilde{u}|^2 dx. \tag{3.4}$$

Fixing ϵ such that $1 - 8C\epsilon^2 > 0$, we obtain

$$\sum_{|\alpha|\le 2} \beta^{2-2|\alpha|} \int |x|^{-2\beta+2|\alpha|-3}|\partial_x^\alpha \tilde{u}|^2 dx \le C \int |x|^{-2\beta+1}|P(x,\partial)\tilde{u}|^2 dx \tag{3.5}$$

by (3.2), (3.3) and (3.4). On the other hand, from (2.4) and $\epsilon\beta^{-1} \le |x| \le 2\epsilon\beta^{-1}$, we have

$$\begin{aligned}\int |x|^{-2\beta+1}|P(x,\partial)\tilde{u}|^2 dx &\le \int_{|x|\le\epsilon\beta^{-1}} |x|^{-2\beta+1}|P(x,\partial)u|^2 dx + \int_{\epsilon\beta^{-1}\le|x|\le 2\epsilon\beta^{-1}} |x|^{-2\beta+1}|P(x,\partial)\tilde{u}|^2 dx \\ &\le 2\int_{|x|\le\epsilon\beta^{-1}} |x|^{-2\beta-1}(C_0^2|x|^{-2}|u|^2 + C_1^2|\nabla u|^2)dx \\ &\quad + C\sum_{|\alpha|\le 2}\int_{\epsilon\beta^{-1}\le|x|\le 2\epsilon\beta^{-1}} |x|^{-2\beta+2|\alpha|-3}|\partial_x^\alpha u|^2 dx.\end{aligned} \tag{3.6}$$

By (3.5) and (3.6), if C_1 is small enough, then we have

$$\sum_{|\alpha|\le 2} \beta^{2-2|\alpha|} \int |x|^{-2\beta+2|\alpha|-3}|\partial_x^\alpha \tilde{u}|^2 dx \le C\sum_{|\alpha|\le 2}\int_{\epsilon\beta^{-1}\le|x|\le 2\epsilon\beta^{-1}} |x|^{-2\beta+2|\alpha|-3}|\partial_x^\alpha u|^2 dx$$

for any large $\beta \in \{j+1/2; j\in\mathbf{N}\}$. It follows that

$$\begin{aligned}\beta^{-2}(\epsilon\beta^{-1}/2)^{-2\beta+1}\sum_{|\alpha|\le 2}\int_{|x|\le 1/2\epsilon\beta^{-1}} |\partial_x^\alpha u|^2 dx &\le \sum_{|\alpha|\le 2}\beta^{2-2|\alpha|}\int_{|x|\le 1/2\epsilon\beta^{-1}} |x|^{-2\beta+2|\alpha|-3}|\partial_x^\alpha \tilde{u}|^2 dx \\ &\le \sum_{|\alpha|\le 2}\beta^{2-2|\alpha|}\int |x|^{-2\beta+2|\alpha|-3}|\partial_x^\alpha \tilde{u}|^2 dx\end{aligned}$$

and

$$\sum_{|\alpha|\le 2}\int_{\epsilon\beta^{-1}\le|x|\le 2\epsilon\beta^{-1}} |x|^{-2\beta+2|\alpha|-3}|\partial_x^\alpha u|^2 dx \le (\epsilon\beta^{-1})^{-2\beta-3}\sum_{|\alpha|\le 2}\int_{\epsilon\beta^{-1}\le|x|\le 2\epsilon\beta^{-1}} |\partial_x^\alpha u|^2 dx.$$

Therefore there exist positive C_3 and C_4 such that

$$\begin{aligned}\sum_{|\alpha|\le 2}\int_{|x|\le 1/2\epsilon\beta^{-1}} |\partial_x^\alpha u|^2 dx &\le C(1/2)^{2\beta-1}\beta^6\sum_{|\alpha|\le 2}\int_{\epsilon\beta^{-1}\le|x|\le 2\epsilon\beta^{-1}} |\partial_x^\alpha u|^2 dx \\ &\le C_3\exp(-C_4\beta),\end{aligned}$$

which proves the desired conclusion. □

Proposition 3.3 allows us to use $e^{\gamma/2(\log|x|)^2}$ rather than the usual weight $|x|^{-\gamma}$. Using the similar method as [3], we can get the following Carleman estimate under the hypothesis of Theorem 2.1.

Proposition 3.4. *Under the hypothesis of Theorem* 2.1 *there exists a positive C such that*

$$\gamma \||x|\varphi_\gamma \nabla u|x|^{-n/2}\|^2 + \gamma^3 \|\varphi_\gamma u|x|^{-n/2}\|^2 \leq C\||x|^2 \varphi_\gamma P(x,\partial)u|x|^{-n/2}\|^2 \tag{3.7}$$

for any large γ and any $u \in C_0^\infty(\tilde{\Omega} \setminus \{0\})$ with a sufficiently small $\tilde{\Omega}$ where $\varphi_\gamma = \exp(\gamma/2(\log|x|)^2)$.

Remark 3.5. The estimate (3.7) in Proposition 3.4 remains valid if we assume $u \in H^2_{loc}(\Omega)$ with compact support satisfies

$$\lim_{\rho\to 0} \exp(\beta(\log|\rho|)^2) \int_{|x|\leq\rho} |\partial_x^\alpha u|^2 dx = 0$$

for any positive β and any $|\alpha| \leq 2$.

Proof. Let's introduce polar coordinates in $\mathbf{R}^n \setminus \{0\}$ by setting $x = e^t\omega$, with $t \in \mathbf{R}$ and $\omega = (\omega_1, \ldots, \omega_n) \in S^{n-1}$. For $k = 1, \ldots, n$, we set $D_k = \Omega_k$ and $D_0 = \partial_t$. We have then

$$\partial_{x_j} = e^{-t}(\omega_j \partial_t + \Omega_j)$$

where Ω_j is a vector field on S^{n-1}. The vector fields Ω_j have the properties

$$\sum_{j=1}^n \omega_j \Omega_j = 0 \quad \text{and} \quad \sum_{j=1}^n \Omega_j \omega_j = n-1.$$

The adjoint of Ω_j as an operator in $L^2(S^{n-1})$ is

$$\Omega_j^* = (n-1)\omega_j - \Omega_j.$$

Then the operator $P(x,\partial)$ takes the form

$$\begin{aligned} e^{2t}P(x,\partial) &= e^{2t}P(0,\partial) + e^{2t}(P(x,\partial) - P(0,\partial)) \\ &= (\partial_t^2 + (n-2)\partial_t + \triangle_\omega)I_N \\ &\quad + \sum_{j,k} (A_{j,k}(e^t\omega) - A_{j,k}(0)) \\ &\quad \times (\omega_j(\partial_t - 1) + \Omega_j)(\omega_k \partial_t + \Omega_k) \\ &:= (\partial_t^2 + (n-2)\partial_t + \triangle_\omega)I_N \\ &\quad + \sum_{j+|\alpha|\leq 2} B_{j,\alpha}(t,\omega)\partial_t^j \Omega^\alpha, \end{aligned}$$

where $B_{j,\alpha}$ are $N \times N$ valued matrices such that $B_{j,\alpha} = O(e^t)$ as t tends to $-\infty$, and $D_k B_{j,\alpha} = O(e^t)$ as t tends to $-\infty$, for any $k \in \{0, 1, \ldots, n\}$. We note

$$B_{j,\alpha}^* B_{k,\beta} = B_{k,\beta} B_{j,\alpha}^* \tag{3.8}$$

for any j, k, α and β thanks to the hypothesis (2.2). Set $u = e^{-\gamma t^2/2}v$ and $P_\gamma v = e^{\gamma t^2/2}P(e^{-\gamma t^2/2}v)$, thus P_γ can be written

$$\begin{aligned} e^{2t}P_\gamma v &= (\partial_t - \gamma t)^2 v + (n-2)(\partial_t - \gamma t)v + \triangle_\omega v \\ &\quad + \sum_{j+|\alpha|\leq 2} B_{j,\alpha}(\partial_t - \gamma t)^j \Omega^\alpha v \\ &= \partial_t^2 v + (n - 2 - 2\gamma t)\partial_t v \\ &\quad + (\gamma^2 t^2 - \gamma - (n-2)\gamma t + \triangle_\omega)v \\ &\quad + \sum_{j+|\alpha|\leq 2} B_{j,\alpha}(\partial_t - \gamma t)^j \Omega^\alpha v \\ &= a_1 + a_2 + a_3 + a_4. \end{aligned}$$

Set

$$\begin{aligned} e^{2t}P_\gamma^- v &= (-\partial_t - \gamma t)^2 v + (n-2)(-\partial_t - \gamma t)v \\ &\quad + \triangle_\omega v - 2\gamma v \\ &\quad + \sum_{j+|\alpha|\leq 2} B_{j,\alpha}^*(-\partial_t - \gamma t)^j (\Omega^*)^\alpha v \\ &= \partial_t^2 v - (n - 2 - 2\gamma t)\partial_t v \\ &\quad + (\gamma^2 t^2 - \gamma - (n-2)\gamma t + \triangle_\omega)v \\ &\quad + \sum_{j+|\alpha|\leq 2} B_{j,\alpha}^*(-\partial_t - \gamma t)^j (\Omega^*)^\alpha v \\ &= a_1 - a_2 + a_3 + a_5, \end{aligned}$$

$$D(\gamma, v) = \|e^{2t}P_\gamma v\|^2 - \|e^{2t}P_\gamma^- v\|^2$$

and

$$S(\gamma, v) = \|e^{2t}t^{-1}P_\gamma v\|^2 + \|e^{2t}t^{-1}P_\gamma^- v\|^2.$$

Thus we have

$$\begin{aligned} D(\gamma, v) &= 4\Re\{(a_1, a_2) + (a_2, a_3)\} \\ &\quad + 2\Re\{(a_4, a_1 + a_2 + a_3) - (a_1 - a_2 + a_3, a_5)\} \\ &\quad + \|a_4\|^2 - \|a_5\|^2 \\ S(\gamma, v) &\geq \|t^{-1}(a_1 + a_2 + a_3)\|^2/2 \\ &\quad + \|t^{-1}(a_1 - a_2 + a_3)\|^2/2 \\ &\quad - \|t^{-1}a_4\|^2 - \|t^{-1}a_5\|^2 \\ &= \|t^{-1}a_1\|^2 + \|t^{-1}a_2\|^2 + \|t^{-1}a_3\|^2 \\ &\quad + 2\Re(t^{-1}a_1, t^{-1}a_3) - \|t^{-1}a_4\|^2 - \|t^{-1}a_5\|^2 \end{aligned}$$

where $(\cdot,\cdot)$ is the L^2 inner product. Using the same method as [3], we have

$$2\Re(a_1, a_2) = 2\gamma\|\partial_t v\|^2,$$

$$2\Re(a_2, a_3) = \|fv\|^2/2 - 2\gamma \sum_{j=1}^n \|\Omega_j v\|^2,$$

$$\begin{aligned} &\|t^{-1}a_1\|^2 + \|t^{-1}a_2\|^2 + \|t^{-1}a_3\|^2 + 2\Re(t^{-1}a_1, t^{-1}a_3) \\ &\quad = \|t^{-1}\partial_t^2 v\|^2 + \|t^{-1}\triangle_\omega v\|^2 + 2\sum_{j=1}^n \|t^{-1}\partial_t \Omega_j v\|^2 \end{aligned}$$

$$+ \|hv\|^2 + \|g\partial_t v\|^2 - \sum_{j=1}^{n} \|l\Omega_j v\|^2$$

where $f^2 = 12\gamma^3 t^2 - 12(n-2)\gamma^2 t - 4\gamma^2 + 2(n-2)^2\gamma$, $g^2 = (-2\gamma + (n-2)t^{-1})^2 - 2\gamma^2 + 2(n-2)\gamma t^{-1} + 2\gamma t^{-2}$, $h^2 = (\gamma^2 t - (n-2)\gamma - \gamma t^{-1})^2 - 2(n-2)\gamma t^{-3} - 6\gamma t^{-4}$ and $l^2 = 2(\gamma^2 - (n-2)\gamma t^{-1} - \gamma t^{-2}) + 6t^{-4}$ (see from page 212, line 23 to page 213, line 18 in [3] in detail). On the other hand, by the definition of a_4 and a_5 it follows that

$$\|t^{-1}a_4\| + \|t^{-1}a_5\| \le \sum_{|\alpha|\le 2} \gamma^{4-2|\alpha|} \|B_1 D^\alpha v\|^2$$

where $B_1 = B_1(t,\omega)$ satisfies $B_1(t,\omega) = O(te^t)$ as t tends to $-\infty$.

To prove Proposition 3.4 we need the following similar result as [3] (see Lemma 2.3 in [3]).

Lemma 3.6. *Let $\tilde{D}_0 = \partial_t - \gamma t$ and $\tilde{D}_j = \Omega_j$ for $j = 1,\dots,n$, and let $A(t,\omega)$ be an $N \times N$ matrix valued function such that $A = O(e^t)$ as t tends to $-\infty$, and $D_k A = O(e^t)$ as t tends to $-\infty$. Then there exists $B(t,\omega)$ such that for any $v \in C_0^\infty(I \times S^{n-1})$, and for any $\alpha, \beta \in \mathbf{N}^{n+1}$, with $|\alpha|, |\beta| \le 2$ we have*

$$(A(t,\omega)\tilde{D}^\alpha v, \tilde{D}^\beta v) - (A(t,\omega)(\tilde{D}^*)^\beta v, (\tilde{D}^*)^\alpha v) \le \sum_{|\alpha|\le 2} \gamma^{3-2|\alpha|} \|BD^\alpha v\|^2$$

and $B = O(t^2 e^{t/2})$ as t tends to $-\infty$.

Now, we proceed to the proof of Proposition 3.4. By Lemma 3.6 and (3.8) there exists a function $B_2(t,\omega)$ such that $B_2(t,\omega) = O(t^2 e^{t/2})$ as t tends to $-\infty$ and

$$\begin{aligned} &2\Re\{(a_4, a_1 + a_2 + a_3) - (a_1 - a_2 + a_3, a_5)\} \\ &\quad + \|a_4\|^2 - \|a_5\|^2 \\ &\le \sum_{|\alpha|\le 2} \gamma^{3-2\alpha} \|B_2 D^\alpha v\|^2. \end{aligned}$$

Thus we have

$$D(\gamma, v) \ge 4\gamma\|\partial_t v\|^2 + \|fv\|^2 - 4\gamma \sum_{j=1}^{n} \|\Omega_j v\|^2 - \sum_{|\alpha|\le 2} \gamma^{3-2\alpha} \|B_2 D^\alpha v\|^2$$

and

$$\begin{aligned} S(\gamma, v) &\ge \|t^{-1}\partial_t^2 v\|^2 + 2\sum_{1}^{n} \|t^{-1}\partial\Omega_j v\|^2 \\ &\quad + \|t^{-1}\triangle_\omega v\|^2 + \|g\partial_t v\|^2 - \sum_{j=1}^{n} \|l\Omega_j v\|^2 \\ &\quad + \|hv\|^2 - \sum_{|\alpha|\le 2} \gamma^{4-2|\alpha|} \|B_1 D^\alpha v\|^2. \end{aligned}$$

Therefore we have

$$\begin{aligned} &\gamma D(\gamma, v) + S(\gamma, v) \\ &\quad \ge \|t^{-1}\partial_t^2 v\|^2 + 2\sum_{j=1}^{n} \|t^{-1}\partial_t \Omega_j v\|^2 + \|t^{-1}\triangle_\omega v\|^2 \\ &\qquad + 4\gamma^2 \|\partial_t v\|^2 + \|g\partial_t v\|^2 + \|hv\|^2 + \gamma\|fv\|^2 \\ &\qquad - \sum_{j=1}^{n} \|l\Omega_j v\|^2 - 4\gamma^2 \sum_{j=1}^{n} \|\Omega_j v\|^2 \\ &\qquad - \sum_{|\alpha|\le 2} \gamma^{4-2|\alpha|} \|BD^\alpha v\|^2 \end{aligned}$$

where $B = O(t^2 e^{t/2})$ as t tends to $-\infty$. Using the same method in [3], if $|T_0|$ is sufficiently large, then we conclude that there exists a positive constant C such that

$$\|e^{2t} P_\gamma v\|^2 \ge C \sum_{|\alpha|\le 2} \gamma^{3-2|\alpha|} \|t^{1-|\alpha|} D^\alpha v\|^2$$

for any $v \in C_0^\infty((-\infty, T_0) \times S^{n-1})$, which is a better estimate than the desired one (3.7) (see from page 214, line 2 to page 215, line 12 in [3]). □

By Proposition 3.3 and Proposition 3.4 we can see Theorem 2.1 in the standard manner.

Acknowledgements. The author would like to express his sincere gratitude to Prof. Tatsuo Nishitani for his valuable advice, support and constant encouragement. The author also thank the referee for a very careful reading of my manuscript.

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