Multiple Dedekind sums attached to Dirichlet characters

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Abstract: We study many-term relations for multiple Dedekind sums including the case generalized by means of Dirichlet characters. We mainly use the method due to Carlitz. The main results contain the original reciprocity formula for Dedekind sums and its generalized formulas due to Apostol, Snyder and Carlitz.

Key words: Multiple Dedekind sum; reciprocity formula.

1. Introduction. For any rational number x, we denote by $[x]$ the greatest integer not exceeding x and put

$$((x)) = \begin{cases} x - [x] - \dfrac{1}{2} & \text{if } x \text{ is not an integer} \\ 0 & \text{otherwise.} \end{cases}$$

For integers h and k with $k > 0$, the classical Dedekind sum $s(h,k)$ is defined by

$$s(h,k) = \sum_{\mu \bmod k} \left(\left(\frac{\mu}{k}\right)\right)\left(\left(\frac{h\mu}{k}\right)\right).$$

If $h, k > 0$ with $(h,k) = 1$, the original reciprocity formula for Dedekind sums is expressed by

$$(1) \quad 12hk\{s(h,k) + s(k,h)\} = h^2 - 3hk + k^2 + 1.$$

Generalizations of this formula have been studied extensively. For any integer $m \geq 0$, let $B_m(X)$ be the mth Bernoulli polynomial and put $\tilde{B}_m(x) = B_m(x - [x])$. In [1], Apostol defined the higher order Dedekind sums by

$$s_m(h,k) = \sum_{r \bmod k} \tilde{B}_1\left(\frac{r}{k}\right)\tilde{B}_m\left(\frac{hr}{k}\right)$$

and he generalized the formula (1) as

$$\begin{aligned} (2) \quad (m+1)\{hk^m s_m(h,k) + kh^m s_m(k,h)\} &= \sum_{j=0}^{m+1} \binom{m+1}{j} B_j B_{m+1-j} h^j k^{m+1-j} + mB_{m+1} \\ &= (hB + kB)^{m+1} + mB_{m+1} (\text{symbolically}), \end{aligned}$$

where B_m is the mth Bernoulli number. In [3], Carlitz gave another proof of (2) by making use of certain polynomial identities.

Generalizations of (2) by means of Dirichlet characters are studied in Nagasaka [8] and Snyder [10]. Especially, an explicit generalization of (2) is stated in [10] as follows:

Let χ be a primitive Dirichlet character with conductor f_χ and define

$$(3) \quad \tilde{B}_{m,\chi}(x) = f_\chi^{m-1} \sum_{j \bmod f_\chi} \chi(x+j)\tilde{B}_m\left(\frac{x+j}{f_\chi}\right)$$

for any $x \in \mathbf{Q}$ with denominator relatively prime to f_χ and

$$s_{m,\chi}(h,k) = \sum_{r \bmod k} \tilde{B}_1\left(\frac{r}{k}\right)\tilde{B}_{m,\chi}\left(\frac{hr}{k}\right).$$

Then, we have

$$\begin{aligned} (4) \quad (m+1)\{hk^m\chi(k)s_{m,\chi}(h,k) + kh^m\chi(h)s_{m,\chi}(k,h)\} &= f_\chi^{m-1} \sum_{0 \leq i,j \leq f_\chi - 1} \chi(hi + kj) \\ &\times \left\{h\tilde{B}\left(\frac{i}{f_\chi}\right) + k\tilde{B}\left(\frac{j}{f_\chi}\right)\right\}^{m+1} + mB_{m+1,\chi}, \end{aligned}$$

where $B_{m,\chi}$ is the generalized mth Bernoulli number.

Other generalizations of (1) and (2) are studied in [4] and [5], where Carlitz defined multiple Dedekind sums and derived many-term relations.

In this paper, we first give a definition of multiple Dedekind sums, which is essentially a natural generalization of that in [4] but somewhat different from that in [5]. We also extend the definition by means of Dirichlet characters. Then, extending the method in [4] and [5], we derive

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formulas including (2), (4) and Theorem 1 of [4] as special cases.

Throughout the paper, we denote by $\mathbf{Q}$, $\mathbf{Z}$ and $\mathbf{N}$, the rational number field, the ring of integers of $\mathbf{Q}$ and the set of positive integers, respectively, as usual.

2. Definition of multiple Dedekind sums. As in Introduction, let B_m and $B_m(X)$ be the mth Bernoulli number and polynomial, respectively, defined by

$$\frac{t}{e^t-1}=\sum_{m=0}^{\infty}B_m\frac{t^m}{m!}\quad\text{and}\quad\frac{te^{tX}}{e^t-1}=\sum_{m=0}^{\infty}B_m(X)\frac{t^m}{m!}.$$

We put $\tilde{B}_m(x)=B_m(x-[x])$ for any $x\in\mathbf{Q}$.

Let $a_1,\cdots,a_n\in\mathbf{Z}$ and set $A=(a_1,\cdots,a_n)$. For any $k,m\in\mathbf{N}$, we define multiple Dedekind sums $s_m^{(n)}(A;k)=s_m^{(n)}(a_1,\cdots,a_n;k)$ by

$$s_m^{(n)}(A;k)=\sum_{r_1,\cdots,r_n \bmod k}\left(\prod_{i=1}^{n}\tilde{B}_1\left(\frac{r_i}{k}\right)\right)\times\tilde{B}_m\left(\frac{a_1r_1+\cdots+a_nr_n}{k}\right).$$

Let χ be a primitive Dirichlet character with conductor f_χ and $B_{m,\chi}$ the generalized mth Bernoulli number defined by

$$\sum_{j=0}^{f_\chi-1}\frac{\chi(j)te^{jt}}{e^{f_\chi t}-1}=\sum_{m=0}^{\infty}B_{m,\chi}\frac{t^m}{m!}.$$

For any $x\in\mathbf{Q}$ with denominator relatively prime to f_χ, let $\tilde{B}_{m,\chi}(x)$ be as (3). We define multiple Dedekind sums attached to χ by

$$s_{m,\chi}^{(n)}(A;k)=\sum_{r_1,\cdots,r_n \bmod k}\left(\prod_{i=1}^{n}\tilde{B}_1\left(\frac{r_i}{k}\right)\right)\times\tilde{B}_{m,\chi}\left(\frac{a_1r_1+\cdots+a_nr_n}{k}\right).$$

3. Expressions by Euler numbers. As in [7], we define the modified Euler numbers $E_m(u)$ belonging to a parameter u by

$$\frac{u}{e^z-u}=\frac{E_{-1}(u)}{z}+\sum_{m=0}^{\infty}E_m(u)\frac{z^m}{m!}. \tag{5}$$

Note that $E_{-1}(u)\neq 0$ only if $u=1$ and that we have $E_{-1}(1)=1$ and $mE_{m-1}(1)=B_m$ for all $m\in\mathbf{N}$. It is known that for any $a\in\mathbf{Z}$, $k,m\in\mathbf{N}$ and for any kth root of unity ξ, we have

$$mE_{m-1}(\xi)=k^{m-1}\sum_{j=0}^{k-1}\tilde{B}_m\left(\frac{j}{k}\right)\xi^{-j}. \tag{6}$$

and

$$k^m\tilde{B}_m\left(\frac{a}{k}\right)=m\sum_{\zeta^k=1}E_{m-1}(\zeta)\zeta^a$$

(Section 6 of [2]). Hence direct calculation shows that

$$k^m s_m^{(n)}(A;k)=m\sum_{\zeta^k=1}\left(\prod_{i=1}^{n}E_0(\zeta^{-a_i})\right)E_{m-1}(\zeta), \tag{7}$$

which is equivalent to (3.2) of [4] if $m=1$. (We note here that (3.2) of [4] is to be corrected by replacing $1/k^n$ by $1/k$ in the right-hand side, so that Theorem 1 of [4] also to be corrected by removing k_n^{n-2}.)

Let $E_{m,\chi}(u)$ be the numbers (a modification of the generalized Euler numbers in [11]) defined by

$$\sum_{j=0}^{f_\chi-1}\frac{\chi(j)u^{f_\chi-j}e^{jz}}{e^{f_\chi z}-u^{f_\chi}}=\frac{E_{-1,\chi}(u)}{z}+\sum_{m=0}^{\infty}E_{m,\chi}(u)\frac{z^m}{m!}.$$

Note that $E_{-1,\chi}(u)\neq 0$ only if u is a primitive f_χth root of unity. Note also that $mE_{m-1,\chi}(1)=B_{m,\chi}$ for all $m\in\mathbf{N}$. Let ζ_χ be an arbitrary primitive f_χth root of unity and put $\tau(\chi,\zeta_\chi)=\sum_{j=0}^{f_\chi-1}\chi(j)\zeta_\chi^j$, the Gauss sum attached to χ and ζ_χ. Then

$$\sum_{j=0}^{f_\chi-1}\frac{\chi(j)u^{f_\chi-j}e^{jz}}{e^{f_\chi z}-u^{f_\chi}}=\frac{\tau(\chi,\zeta_\chi)}{f_\chi}\sum_{j=0}^{f_\chi-1}\chi^{-1}(j)\frac{\zeta_\chi^j u}{e^z-\zeta_\chi^j u},$$

so that

$$E_{m,\chi}(u)=\frac{\tau(\chi,\zeta_\chi)}{f_\chi}\sum_{j=0}^{f_\chi-1}\chi^{-1}(j)E_m(\zeta_\chi^j u). \tag{8}$$

Hence, if $(k,f_\chi)=1$, we deduce

$$\chi(k)k^m\tilde{B}_{m,\chi}\left(\frac{a}{k}\right)=m\sum_{\zeta^k=1}E_{m-1,\chi}(\zeta)\zeta^a$$

((3.8) of [6]) and as a generalization of (7) we obtain

$$\chi(k)k^m s_{m,\chi}^{(n)}(A;k)=m\sum_{\zeta^k=1}\left(\prod_{i=1}^{n}E_0(\zeta^{-a_i})\right)E_{m-1,\chi}(\zeta). \tag{9}$$

4. Identities of rational functions. In this section, making use of the method in [4] and [5], we deduce some identities of rational functions concerning multiple Dedekind sums.

Let $a_1,\cdots,a_n\in\mathbf{N}$ be relatively prime in pairs and set $A=(a_1,\cdots,a_n)$ and $A_i=(a_1,\cdots,a_{i-1},$

$a_{i+1},\cdots,a_n)$ for $i=1,\cdots,n$. We put

$$G(X;A)=\prod_{i=1}^{n}\frac{X^{a_i}-1}{X-1}.$$

and

$$(10)\qquad F(X;A)=\sum_{i=1}^{n}\frac{1}{a_i}\sum_{\zeta_i}{}'\frac{1}{G(\zeta_i;A_i)}\frac{\zeta_i}{X-\zeta_i}$$
$$=\sum_{i=1}^{n}\frac{1}{a_i}\sum_{\zeta_i}{}'\frac{(\zeta_i-1)^{n-1}}{\prod_{j\neq i}(\zeta_i^{a_j}-1)}\frac{\zeta_i}{X-\zeta_i},$$

where the symbol $\sum_{\zeta_i}'$ means to take sum over all the a_ith root of unity distinct from 1. Then as shown in [4], we have

$$(11)\quad F(X;A)=\frac{1}{(X-1)G(X;A)}-\frac{1}{a_1\cdots a_n(X-1)}$$
$$=\frac{(X-1)^{n-1}}{\prod_{i=1}^{n}(X^{a_i}-1)}-\frac{1}{a_1\cdots a_n(X-1)}$$

((2.5) of [4]). Now we have

$$\frac{1}{(X-1)^{n-1}(X-\alpha)}$$
$$=-\sum_{l=1}^{n-1}\frac{1}{(\alpha-1)^l}\frac{1}{(X-1)^{n-l}}+\frac{1}{(\alpha-1)^{n-1}}\frac{1}{X-\alpha}.$$

Hence we see from (10) that

$$\frac{F(X;A)}{(X-1)^{n-1}}$$
$$=-\sum_{l=1}^{n-1}\left(\sum_{i=1}^{n}\frac{1}{a_i}\sum_{\zeta_i}{}'\frac{1}{G(\zeta_i;A_i)}\frac{\zeta_i}{(\zeta_i-1)^l}\right)\frac{1}{(X-1)^{n-l}}$$
$$+\sum_{i=1}^{n}\frac{1}{a_i}\sum_{\zeta_i}{}'\frac{1}{\prod_{j\neq i}(\zeta_i^{a_j}-1)}\frac{\zeta_i}{X-\zeta_i}.$$

Note that

$$F^{(l-1)}(X;A)$$
$$=-(l-1)!\sum_{i=1}^{n}\frac{1}{a_i}\sum_{\zeta_i}{}'\frac{1}{G(\zeta_i;A_i)}\frac{\zeta_i}{(\zeta_i-X)^l}$$

for any $l\in\mathbf{N}$ and

$$E_0(\zeta_i^{-a_j})=\frac{1}{\zeta_i^{a_j}-1}.$$

It follows that

$$(12)\qquad \frac{F(X;A)}{(X-1)^{n-1}}=\sum_{l=1}^{n-1}\frac{F^{(l-1)}(1;A)}{(l-1)!}\frac{1}{(X-1)^{n-l}}$$
$$+\sum\frac{1}{a_i}\sum_{\zeta_i}{}'\left(\prod_{j\neq i}E_0(\zeta_i^{-a_j})\right)\frac{\zeta_i}{X-\zeta_i}.$$

We put

$$(13)\qquad H(X;A)=\sum_{i=1}^{n}\frac{1}{a_i}\sum_{\zeta_i}{}'\left(\prod_{j\neq i}E_0(\zeta_i^{-a_j})\right)\frac{\zeta_i}{X-\zeta_i}.$$

As in Section 4 of [4], let $Z_m^{(n)}(A)$ be the number defined by

$$F(1+t;A)=\frac{t^{n-1}}{\prod_{i=1}^{n}((1+t)^{a_i}-1)}-\frac{1}{a_1\cdots a_n t}$$
$$=\sum_{m=1}^{\infty}Z_m^{(n)}(A)\frac{t^{m-1}}{m!}.$$

Then, we have

$$F^{(m-1)}(1;A)=\frac{Z_m^{(n)}(A)}{m}.$$

(For more precise description of $Z_m^{(n)}(A)$, see [4].) Hence we see from (11), (12) and (13) that

$$(14)\qquad H(1+t;A)=\frac{1}{\prod_{i=1}^{n}((1+t)^{a_i}-1)}-\frac{1}{a_1\cdots a_n t^n}$$
$$-\sum_{l=1}^{n-1}\frac{Z_l^{(n)}(A)}{l!}\frac{1}{t^{n-l}}$$
$$(15)\qquad =\sum_{m=n}^{\infty}Z_m^{(n)}(A)\frac{t^{m-n}}{m!}.$$

5. Formulas for $s_m^{(n)}(A;k)$. Replacing X by e^z in (13), we see from (5) that

$$H(e^z;A)$$
$$=\sum_{m=1}^{\infty}\sum_{i=1}^{n}\frac{1}{a_i}\sum_{\zeta_i}{}'\left(\prod_{j\neq i}E_0(\zeta_i^{-a_j})\right)E_{m-1}(\zeta_i)\frac{z^{m-1}}{(m-1)!}.$$

Then from (7) we have

$$(16)\qquad H(e^z;A)=\sum_{m=1}^{\infty}\sum_{i=1}^{n}\Bigg(a_i^{m-1}s_m^{(n-1)}(A_i;a_i)$$
$$-\frac{1}{a_i}\left(-\frac{1}{2}\right)^{n-1}B_m\Bigg)\frac{z^{m-1}}{m!}.$$

Comparing the constant terms of (15) and (16), we obtain

$$\sum_{i=1}^{n} s_1^{(n-1)}(A_i;a_i) = \left(-\frac{1}{2}\right)^n \sum_{i=1}^{n} \frac{1}{a_i} + \frac{Z_n^{(n)}}{n!},$$

which is equivalent to Theorem 1 of [4].

As in Norlund [9], we define the Bernoulli number of higher order $B_m^{(n)}(A) = B_m^{(n)}(a_1, \cdots, a_n)$ by

$$(17) \qquad \frac{a_1 \cdots a_n z^n}{\prod_{i=1}^{n}(e^{a_i z} - 1)} = \sum_{m=0}^{\infty} B_m^{(n)}(A) \frac{z^m}{m!}.$$

We put $B_m^{(n)} = B_m^{(n)}(1, \cdots, 1)$. Then

$$B_m^{(n)}(a_1, \cdots, a_n) = \sum_{\substack{\nu_1, \cdots, \nu_n \geq 0 \\ \nu_1 + \cdots + \nu_n = m}} \frac{m!}{\nu_1! \cdots \nu_n!} a_1^{\nu_1} \cdots a_n^{\nu_n} B_{\nu_1} \cdots B_{\nu_n} = (a_1 B + \cdots + a_n B)^m.$$

From (14) and (17), we see that

$$(18) \qquad H(e^z; A) = \frac{1}{a_1 \cdots a_n} \sum_{m=0}^{\infty} (B_m^{(n)}(A) - B_m^{(n)}) \frac{z^{m-n}}{m!} - \sum_{l=1}^{n-1} \frac{Z_l^{(n)}(A)}{l!} \sum_{m=0}^{\infty} B_m^{(n-l)} \frac{z^{m-n+l}}{m!}.$$

Comparing the coefficients of (16) and (18), we obtain the following

Theorem 1. *Let $a_1, \cdots, a_n \in \mathbf{N}$ be relatively prime in pairs and set $A = (a_1, \cdots, a_n)$ and $A_i = (a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n)$. Then for any $n, m \in \mathbf{N}$ with $n \geq 2$, we have*

$$\sum_{i=1}^{n} a_i^{m-1} s_m^{(n-1)}(A_i; a_i) = \left(-\frac{1}{2}\right)^{n-1} B_m \sum_{i=1}^{n} \frac{1}{a_i} + \frac{m!}{a_1 \cdots a_n (m+n-1)!} (B_{m+n-1}^{(n)}(A) - B_{m+n-1}^{(n)}) - \sum_{l=1}^{n-1} \frac{Z_l^{(n)}(A)}{l!} \frac{m! B_{m+n-1-l}^{(n-l)}}{(m+n-1-l)!}.$$

Remark. For $n = 2$, it is easy to verify that

$$Z_1^{(2)}(a_1, a_2) = -\frac{1}{2}\left(\frac{1}{a_1} + \frac{1}{a_2}\right) + \frac{1}{a_1 a_2}$$

and

$$\frac{1}{m+1} B_{m+1}^{(2)} = -B_m - \frac{m}{m+1} B_{m+1},$$

so that Theorem 1 reduces to Apostol's result (2).

6. Formulas for $s_{m,\chi}^{(n)}(A;k)$. Let χ be a non-trivial primitive Dirichlet character with conductor f_χ and ζ_χ an arbitrary primitive f_χth root of unity. We assume that $(a_1 \cdots a_n, f_\chi) = 1$. We define the numbers $B_{m,\chi}^{(n)}(A) = B_{m,\chi}^{(n)}(a_1, \cdots, a_n)$ by

$$(19) \qquad \frac{\tau(\chi, \zeta_\chi)}{f_\chi} \sum_{j=0}^{f_\chi - 1} \chi^{-1}(j) \frac{a_1 \cdots a_n z^n}{\prod_{i=1}^{n}(\zeta_\chi^{-j a_i} e^{a_i z} - 1)} = \sum_{m=0}^{\infty} B_{m,\chi}^{(n)}(A) \frac{z^m}{m!}.$$

Note that the left-hand side of (19) is independent of the choice of ζ_χ. We put $B_{m,\chi}^{(n)} = B_{m,\chi}^{(n)}(1, \cdots, 1)$. Then we have $B_{m,\chi}^{(1)} = B_{m,\chi}$.

Let us deduce an expression of $B_{m,\chi}^{(n)}(A)$ in terms of the Bernoulli numbers. From (5), the left-hand side of (19) equals

$$\frac{\tau(\chi, \zeta_\chi)}{f_\chi} \sum_{j=0}^{f_\chi - 1} \chi^{-1}(j) \prod_{i=1}^{n} \sum_{\nu_i = 0}^{\infty} E_{\nu_i}(\zeta_\chi^{j a_i}) \frac{(a_i z)^{\nu_i + 1}}{\nu_i!}$$
$$= \frac{\tau(\chi, \zeta_\chi)}{f_\chi} \sum_{j=0}^{f_\chi - 1} \chi^{-1}(j) \sum_{m=0}^{\infty} \sum_{\substack{\nu_1, \cdots, \nu_n \geq 0 \\ \nu_1 + \cdots + \nu_n = m}} \frac{m!}{\nu_1! \cdots \nu_n!} \times \left(\prod_{i=1}^{n} E_{\nu_i}(\zeta_\chi^{j a_i})\right) \frac{a_1^{\nu_1 + 1} \cdots a_n^{\nu_n + 1} z^{m+n}}{m!}.$$

By (6), this equals

$$\frac{\tau(\chi, \zeta_\chi)}{f_\chi} \sum_{j=0}^{f_\chi - 1} \chi^{-1}(j) \sum_{m=0}^{\infty} \sum_{\substack{\nu_1, \cdots, \nu_n \geq 0 \\ \nu_1 + \cdots + \nu_n = m}} \frac{m!}{\nu_1! \cdots \nu_n!} \times \left\{\prod_{i=1}^{n} \frac{f_\chi^{\nu_i}}{\nu_i + 1} \sum_{l_i = 0}^{f_\chi - 1} \tilde{B}_{\nu_i + 1}\left(\frac{l_i}{f_\chi}\right) \zeta_\chi^{-j a_i l_i}\right\} \times \frac{a_1^{\nu_1 + 1} \cdots a_n^{\nu_n + 1} z^{m+n}}{m!}$$
$$= \frac{\tau(\chi, \zeta_\chi)}{f_\chi} \times \sum_{0 \leq l_1, \cdots, l_n \leq f_\chi - 1} \sum_{j=0}^{f_\chi - 1} \chi^{-1}(j) \zeta_\chi^{-j(a_1 l_1 + \cdots + a_n l_n)} \times \sum_{m=0}^{\infty} f_\chi^m \sum_{\substack{\nu_1, \cdots, \nu_n \geq 0 \\ \nu_1 + \cdots + \nu_n = m}} \frac{m!}{(\nu_1 + 1)! \cdots (\nu_n + 1)!} \times \left(\prod_{i=1}^{n} a_i^{\nu_i + 1} \tilde{B}_{\nu_i + 1}\left(\frac{l_i}{f_\chi}\right)\right) \frac{z^{m+n}}{m!}$$
$$= \sum_{0 \leq l_1, \cdots, l_n \leq f_\chi - 1} \chi(a_1 l_1 + \cdots + a_n l_n) \sum_{m=n}^{\infty} f_\chi^{m-n}$$

$$\times \sum_{\substack{\nu_1,\cdots,\nu_n\geq 1\\ \nu_1+\cdots+\nu_n=m}} \frac{m!}{\nu_1!\cdots\nu_n!}\left(\prod_{i=1}^{n} a_i^{\nu_i}\tilde{B}_{\nu_i}\left(\frac{l_i}{f_\chi}\right)\right)\frac{z^m}{m!}.$$

Thanks to the existence of $\chi(a_1l_1+\cdots+a_nl_n)$ in the last part of the above equation, it makes no difference if we extend the range of $\nu_1,\cdots,\nu_n$ to $\nu_1,\cdots,\nu_n\geq 0$ with $\nu_1+\cdots+\nu_n=m$. Consequently we obtain

$$B_{m,\chi}^{(n)}(A) = \begin{cases} f_\chi^{m-n}\displaystyle\sum_{0\leq l_1,\cdots,l_n\leq f_\chi-1}\chi(a_1l_1+\cdots+a_nl_n) \\ \qquad\times\left\{a_1\tilde{B}\left(\dfrac{l_1}{f_\chi}\right)+\cdots+a_n\tilde{B}\left(\dfrac{l_n}{f_\chi}\right)\right\}^m & \text{if } m\geq n \\ 0 & \text{otherwise.}\end{cases}$$

Now we put

$$H_\chi(X;A)=\frac{\tau(\chi,\zeta_\chi)}{f_\chi}\sum_{j=0}^{f_\chi-1}\chi^{-1}(j)H(\zeta_\chi^{-j}X;A).$$

Then we see from (5), (8), (9) and (13) that

$$H_\chi(e^z;A)=\sum_{m=1}^{\infty}\sum_{i=1}^{n}\left(\chi(a_i)a_i^{m-1}s_{m,\chi}^{(n-1)}(A_i;a_i) - \frac{1}{a_i}\left(-\frac{1}{2}\right)^{n-1}B_{m,\chi}\right)\frac{z^{m-1}}{m!}. \tag{20}$$

On the other hand, from (14) and (19) we deduce

$$H_\chi(e^z;A)=\frac{1}{a_1\cdots a_n}\sum_{m=0}^{\infty}(B_{m,\chi}^{(n)}(A)-B_{m,\chi}^{(n)})\frac{z^{m-n}}{m!} - \sum_{l=1}^{n-1}\frac{Z_l^{(n)}(A)}{l!}\sum_{m=0}^{\infty}B_{m,\chi}^{(n-l)}\frac{z^{m-n+l}}{m!}. \tag{21}$$

Comparing the coefficients of (20) and (21), we obtain the following

Theorem 2. *Let $a_1,\cdots,a_n\in\mathbf{N}$ be relatively prime in pairs and set $A=(a_1,\cdots,a_n)$ and $A_i=(a_1,\cdots,a_{i-1},a_{i+1},\cdots,a_n)$. Let χ be a non-trivial primitive Dirichlet character and assume that $(a_1\cdots a_n,f_\chi)=1$. Then for any $n,m\in\mathbf{N}$ with $n\geq 2$, we have*

$$\sum_{i=1}^{n}\chi(a_i)a_i^{m-1}s_{m,\chi}^{(n-1)}(A_i;a_i)=\left(-\frac{1}{2}\right)^{n-1}B_{m,\chi}\sum_{i=1}^{n}\frac{1}{a_i} + \frac{m!}{a_1\cdots a_n(m+n-1)!}(B_{m+n-1,\chi}^{(n)}(A)-B_{m+n-1,\chi}^{(n)}) - \sum_{l=1}^{n-1}\frac{Z_l^{(n)}(A)}{l!}\frac{m!B_{m+n-1-l,\chi}^{(n-l)}}{(m+n-1-l)!}.$$

Remark. For $n=2$, we have

$$\frac{1}{m+1}B_{m+1,\chi}^{(2)}=-B_{m,\chi}-\frac{m}{m+1}B_{m+1,\chi},$$

so that Theorem 2 reduces to Snyder's formula (4).

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