Projective manifolds with hyperplane sections being four-sheeted covers of projective space

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Abstract: Let L be a very ample line bundle on a smooth complex projective variety X of dimension ≥ 6. We classify the polarized manifolds (X, L) such that there exists a smooth member A of $|L|$ endowed with a branched covering of degree four $\pi\colon A \to \mathbf{P}^n$. The cases of $\deg \pi = 2$ and 3 are already studied by Lanteri-Palleschi-Sommese. Recently the case of $\deg \pi = 5$ is studied by Amitani.

Key words: Polarized variety; hyperplane section; branched covering; linear system; graded ring.

1. Introduction. Let X be an $(n+1)$-dimensional smooth complex projective variety and L a very ample line bundle on X. Consider the following condition:

$(*)_d$ There exists a smooth member $A \in |L|$ such that there exists a finite surjective morphism $\pi\colon A \to \mathbf{P}^n$ of degree d.

Needless to say, the following "obvious" pairs (X, L) satisfy $(*)_d$: $(\mathbf{P}^{n+1}, \mathcal{O}_{\mathbf{P}^{n+1}}(d))$ and $(H_d^{n+1}, \mathcal{O}_{H_d^{n+1}}(1))$, where H_d^{n+1} is a smooth hypersurface of degree d in $\mathbf{P}^{n+2}$.

It is an interesting subject to investigate, for a fixed d, what kind of the "non-obvious" pairs show up. In fact, classical results on surfaces with hyperelliptic curves as hyperplane sections (e.g. [3]) and their revision made in the 1980's called the attention to the problem of classifying pairs (X, L) with $(*)_d$. The problem has been considered by several authors according to the following values of (n, d): $(1, 2)$ (Serrano [15], Sommese-Van de Ven [16]), $(1, 3)$ (Fania [4]).

And, for small prime numbers d, the pairs (X, L) satisfying $(*)_d$ and $n > d$ have been classified completely: For $d = 2$ and 3, Lanteri-Palleschi-Sommese ([11, 12]) classified the pairs. For $d = 5$, Amitani ([1]) classified the pairs recently.

Let q be the morphism associated to $\pi^*\mathcal{O}_{\mathbf{P}^n}(1)$, and assume $t := h^0(A, \pi^*\mathcal{O}_{\mathbf{P}^n}(1)) - n - 1 > 0$. Then we have a factorization of π as follows:

$$\begin{array}{ccl} A & \xrightarrow{\ q\ } & q(A) \subset \mathbf{P}^{n+t} \\ & \underset{\pi}{\searrow} & \Big\downarrow p\colon \text{The projection from a } \mathbf{P}^{t-1} \text{ in } \mathbf{P}^{n+t} \text{ with } q(A) \cap \mathbf{P}^{t-1} = \emptyset. \\ & & \mathbf{P}^n \end{array}$$

In the case where d is a prime, it immediately follows that q is birational onto its image $q(A)$, which is a variety of degree d. This plays a key role in the classification problem for a small d.

Now then, for a composite number d, there may exist pairs (X, L) with a non-birational morphism q. Therefore it is natural to study the structures of these pairs.

The purpose of this article is to give a complete classification of the pairs (X, L) in case $n > d = 4$. Our result is as follows:

Theorem 1.1. *Let X be a smooth projective variety with $\dim X = n + 1 > 5$. Then there exists a very ample line bundle L on X that satisfies the condition $(*)_4$ if and only if (X, L) is one of the following*

(i) $(\mathbf{P}^{n+1}, \mathcal{O}_{\mathbf{P}^{n+1}}(4))$;
(ii) $(\mathbf{Q}^{n+1}, \mathcal{O}_{\mathbf{Q}^{n+1}}(2))$, *where $\mathbf{Q}^{n+1}$ is a smooth hyperquadric in $\mathbf{P}^{n+2}$*;
(iii) $(H_4^{n+1}, \mathcal{O}_{H_4^{n+1}}(1))$;
(iv) $\left(V_{2,2}^{n+1}, \mathcal{O}_{V_{2,2}^{n+1}}(1)\right)$, *where $V_{2,2}^{n+1}$ is a smooth complete intersection of two hyperquadrics in $\mathbf{P}^{n+3}$*;
(v) $(Y, 4\mathcal{L})$, *where $(Y, \mathcal{L})$ is a Del Pezzo manifold of degree one*;
(vi) $(Z, 2\mathcal{L})$, *where $(Z, \mathcal{L})$ is a Del Pezzo manifold*

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of degree 2; *or*

(vii) $(W_{12}, \mathcal{O}_{W_{12}}(4))$, *where* W_{12} *is a smooth hypersurface of degree* 12 *in the weighted projective space* $\mathbf{P}(4, 3, 1^{n+1})$ *with its ample invertible sheaf* $\mathcal{O}_{W_{12}}(1)$.

No less than three "non-obvious" pairs (v)–(vii) show up. The pair (vi) is a unique one with a non-birational morphism q: We see that $q(A)$ is a smooth hyperquadric in this case.

Our basic strategy is to reduce to Fujita's classification theory of polarized varieties, which leads us to study the structure of (X, L) with a non-birational morphism q.

The strategy is roughly summarized as follows: As we will see in the section 3, it follows that $\mathrm{Pic}(X) = \mathbf{Z}[\mathcal{H}]$, where $\mathcal{H}$ is the ample generator. And we can show that invariants of $(X, \mathcal{H})$ are small. Therefore the classification theory is applicable except certain polarized manifolds with sectional genera $g(X, \mathcal{H}) = 3$, Δ-genera and degrees (I) $\Delta(X, \mathcal{H}) = \mathcal{H}^{n+1} = 1$ or (II) 2. The classification problem of polarized manifolds with these invariants, in general, are yet to be solved completely (cf. [7, (6.18), (10.10)]).

As for (I), it turns out that $(X, \mathcal{H})$ is not sectionally hyperelliptic. Furthermore, we find that a curve which is an intersection of n-general members of $|\mathcal{H}|$ is a smooth plane quartic. In this case, we can determine the structure of $(X, \mathcal{H})$ by using a new method developed in [1].

As for (II), we can prove that this case is ruled out by using the Riemann-Roch theorem for curves and the double point formula for surfaces, successfully (see Proposition 3.2).

After the present paper was written up, I found that Antonio Lanteri has obtained a similar classification result in [10, Theorem 3.4] by using the same arguments to rule out some "a priori" possible cases in the section 3 of the present paper. In fact, I found out that Proposition 3.2 and the argument in the proof were the same as [10, Lemma 3.3]. But his classification result contains one doubtful case: In fact, for the case (vii) in Theorem 1.1, his result has given only some invariants. In contrast, our theorem reveals the structure of a unique polarized manifold appearing in the case. So our classification result is complete.

Notations, terminologies and conventions. In this article, we work over the complex number field $\mathbf{C}$. We use the standard notation from algebraic geometry as in [8] and also use the terminologies for polarized varieties as in [7]. For an integer $r \geq 1$, a line bundle L on a manifold M is said to be *r-generated* if the graded ring $R(M, L) := \bigoplus_{i \geq 0} H^0(M, iL)$ is generated by the global sections of $L, \ldots, rL$ (see [9, Definition 2.1]).

2. Three special examples: The 'if' part. In this section we only consider the three special classes (v)–(vii) of polarized manifolds appearing in Theorem 1.1 because one can easily check that the cases (i)–(iv) satisfy the assertion.

Example 1. Let $(X, L) = (Y, 4\mathcal{L})$, where $(Y, \mathcal{L})$ is an $(n+1)$-dimensional Del Pezzo manifold of degree one, i.e., $-K_Y = n\mathcal{L}$ with $\mathcal{L}^{n+1} = 1$. We have $\Delta(Y, \mathcal{L}) = 1$. As in the proof of [12, (1.2)], we see that $4\mathcal{L}$ is very ample. Therefore it follows from [1, Proposition 3.2] that there exists a four-sheeted cover of $\mathbf{P}^n$ that is a member of $|4\mathcal{L}|$.

Example 2. Let $(X, L) = (Z, 2\mathfrak{L})$, where $(Z, \mathfrak{L})$ is an $(n+1)$-dimensional Del Pezzo manifold of degree 2, i.e., $-K_Y = n\mathfrak{L}$ with $\mathfrak{L}^{n+1} = 2$. Then, from [7, (8.11)], $(Z, \mathfrak{L})$ is a double covering of $\mathbf{P}^{n+1}$ branched along a smooth hypersurface of degree 4 and $\mathfrak{L}$ is the pull-back of $\mathcal{O}_{\mathbf{P}^{n+1}}(1)$. The graded ring $R(Z, \mathfrak{L})$ is 2-generated since $(Z, \mathfrak{L})$ is a smooth weighted hypersurface of degree 4 in $\mathbf{P}(2, 1^{n+2})$. We obtain that $2\mathfrak{L}$ is very ample by combining the spannedness of $\mathfrak{L}$ and [9, Corollary 2.3]. Therefore there exists a smooth member $A \in |2\mathfrak{L}|$ that is a double covering of $\mathbf{Q}^n$. By projecting $\mathbf{Q}^n$ from a point of $\mathbf{P}^{n+1} \setminus \mathbf{Q}^n$ to $\mathbf{P}^n$, we see that A is a four-sheeted cover of $\mathbf{P}^n$.

Example 3. Let $(X, L) = (W_{12}, \mathcal{O}_{W_{12}}(4))$, where W_{12} is a smooth weighted hypersurface of degree 12 in $\mathbf{P}(4, 3, 1^{n+1})$. By easy calculations, we obtain that $\Delta(W_{12}, \mathcal{O}_{W_{12}}(1)) = \mathcal{O}_{W_{12}}(1)^{n+1} = 1$. From [6, §13], we see that $\mathrm{Bs}\,|\mathcal{O}_{W_{12}}(1)|$ consists of a single point, which is denoted by p. We obtain a smooth four-sheeted cover of $\mathbf{P}^n$ that is contained in $|\mathcal{O}_{W_{12}}(4)|$ by combining [1, Proposition 3.2] and the following lemma:

Lemma 2.1. *The line bundle* $\mathcal{O}_{W_{12}}(4)$ *is very ample.*

Proof. We obtain the conclusion with the following steps:

(a) $\mathrm{Bs}\,|\mathcal{O}_{W_{12}}(4)| = \emptyset$;

(b) The morphism $\varphi := \varphi_{\mathcal{O}_{W_{12}}(4)}$ associated to $\mathcal{O}_{W_{12}}(4)$ is injective;

(c) The linear system $|\mathcal{O}_{W_{12}}(4)|$ separates the tangent vectors.

From the 4-generatedness of $R(W_{12}, \mathcal{O}_{W_{12}}(1))$ and [9, Theorem 2.2], φ is an embedding outside the single point p. Let x, y, z_j $(0 \leq j \leq n)$ generate the graded ring $R(W_{12}, \mathcal{O}_{W_{12}}(1))$, where $\mathrm{wt}(x, y, z_j) = (4, 3, 1)$ for all j.

(a) It follows that $H^0(\mathcal{O}_{W_{12}}(4))$ is generated by the sections

$$x, yz_0, \ldots, yz_n, z_{j_1} \cdots z_{j_4}, \quad \text{with } 0 \leq j_1 \leq \cdots \leq j_4 \leq n.$$

Therefore we see that

$$\mathrm{Bs}\,|\mathcal{O}_{W_{12}}(4)| = (x = 0) \cap \left(\bigcap_{0 \leq j \leq n} (z_j = 0) \right),$$

which is empty since W_{12} does not meet the singular points of $\mathbf{P}(4, 3, 1^{n+1})$.

(b) If we assume $\varphi(p) = \varphi(q)$ for some $q \in W_{12}$, then we find that $z_j = 0$ for any $0 \leq j \leq n$, which implies $q \in \mathrm{Bs}\,|\mathcal{O}_{W_{12}}(1)|$. Thus $p = q$.

(c) For a non-zero tangent vector $\tau \in T_p(W_{12})$, we need to show that there exists a section $\sigma \in H^0(\mathcal{O}_{W_{12}}(4))$ satisfying the following conditions:

$$\sigma(p) = 0 \text{ and } d\sigma(\tau) \neq 0.$$

We show that $\sigma_j := yz_j$ satisfies the above conditions for some $0 \leq j \leq n$. The former holds because $z_j(p) = 0$ for all j. We prove that the latter holds by contradiction. Assume that there exists a non-zero $\tau \in T_p(W_{12})$ with $d\sigma_j(\tau) = 0$ for all j. Since $d\sigma_j(\tau) = y(p)\, dz_j(\tau)$ and $y(p) \neq 0$, we see that $dz_j(\tau) = 0$ for all j. Thus we have

$$\tau \in T_p(\Gamma), \text{ where } \Gamma := \bigcap_{1 \leq j \leq n} (z_j = 0).$$

It follows from $dz_0(\tau) = 0$ that $\Gamma \cdot \mathcal{O}_{W_{12}}(1) \geq 2$, which contradicts $\mathcal{O}_{W_{12}}(1)^{n+1} = 1$. This completes the proof. □

3. The 'only if' part. Let (X, L) satisfy $n > 4$ and $(*)_4$. And let $\pi\colon A \to \mathbf{P}^n$ denote the finite morphism of degree 4. Then a Barth-type theorem of Lazarsfeld [13, Theorem 1] implies that $H^2(A, \mathbf{Z}) \cong H^2(\mathbf{P}^n, \mathbf{Z}) \cong \mathbf{Z}$ and $H^1(A, \mathcal{O}_A) = 0$. Therefore we have $\mathrm{Pic}(A) \cong \mathbf{Z}$, generated by $\pi^*\mathcal{O}_{\mathbf{P}^n}(1)$. The Lefschetz hyperplane section theorem implies $\mathrm{Pic}(X) \cong \mathbf{Z}$. We denote by $\mathcal{H}$ the ample generator of $\mathrm{Pic}(X)$; we have $\mathcal{H}_A = \pi^*\mathcal{O}_{\mathbf{P}^n}(1)$. Thus we can write $L = l\mathcal{H}$ with some $l > 0$. Since $l\mathcal{H}^{n+1} = \mathcal{H}_A^n = 4$, we see that

$$\mathcal{H}^{n+1} = 1, 2 \text{ or } 4.$$

Combining the ampleness of $\mathcal{H}_A$ and the fact that Δ-genus is non-negative for every polarized manifold [7, Chapter I (4.2)], we see

$$n + 1 \leq h^0(A, \mathcal{H}_A) \leq n + 4.$$

In this section, we investigate the polarized manifolds in question case by case.

The case of $h^0(A, \mathcal{H}_A) = n + 4$. Since $\Delta(A, \mathcal{H}_A) = 0$ and $\mathrm{Pic}(A) \cong \mathbf{Z}$, it follows from [7, Chapter I (5.10)] that $(A, \mathcal{H}_A)$ is either $(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1))$ or $(\mathbf{Q}^n, \mathcal{O}_{\mathbf{Q}^n}(1))$. Moreover, since $\mathcal{H}_A^n = 4$, we get a contradiction. Hence this case does not occur.

The case of $h^0(A, \mathcal{H}_A) = n + 3$. We see that $(A, \mathcal{H}_A)$ has a regular ladder by the argument as in the proof of [1, Lemma 5.1]. Then we obtain that $g(A, \mathcal{H}_A) \geq \Delta(A, \mathcal{H}_A) = 1$ by the Riemann-Roch theorem. Therefore we see $g(A, \mathcal{H}_A) = 1$ by combining $4 = \mathcal{H}_A^n > 2\Delta(A, \mathcal{H}_A) = 2$ and [7, Chapter I (3.5.3)]. This implies that $(A, \mathcal{H}_A)$ is a Del Pezzo manifold of degree 4, which is $\left(V_{2,2}^n, \mathcal{O}_{V_{2,2}^n}(1)\right)$ due to [7, (8.11)].

For $(l, \mathcal{H}^{n+1}) = (1, 4)$, $L = \mathcal{H}$ gives an embedding of X into $\mathbf{P}^{n+3}$. Hence it follows from [14, Corollary 3.8] that $(X, L) \cong \left(V_{2,2}^{n+1}, \mathcal{O}_{V_{2,2}^{n+1}}(1)\right)$. We are in the case (iv) in Theorem 1.1.

For $(l, \mathcal{H}^{n+1}) = (2, 2)$, we see that $h^0(X, \mathcal{H}) = n + 3$ from the Kodaira vanishing theorem. Since $\Delta(X, \mathcal{H}) = 0$ and $\mathcal{H}^{n+1} = 2$, we have $(X, L) \cong (\mathbf{Q}^{n+1}, \mathcal{O}_{\mathbf{Q}^{n+1}}(2))$. Hence we are in the case (ii).

For $(l, \mathcal{H}^{n+1}) = (4, 1)$, we see that this case does not occur as follows: Since $h^0(X, \mathcal{H}) = n + 3$, we obtain that $\Delta(X, \mathcal{H}) = -1$, which is absurd.

The case of $h^0(A, \mathcal{H}_A) = n + 2$. For $(l, \mathcal{H}^{n+1}) = (1, 4)$, we have $h^0(X, \mathcal{H}) = n + 3$ by the Kodaira vanishing theorem. Hence we obtain that $\Delta(X, \mathcal{H}) = 2$. Combining $\dim X > 5$ and [7, (10.8.1)], we see that $(X, L) \cong \left(H_4^{n+1}, \mathcal{O}_{H_4^{n+1}}(1)\right)$. Thus we are in the case (iii) in the Theorem.

For $(l, \mathcal{H}^{n+1}) = (2, 2)$, we have $h^0(X, \mathcal{H}) = n + 2$, hence $\Delta(X, \mathcal{H}) = 1$. It follows from [7, (6.13)] that $(X, L) \cong (Z, 2\mathcal{L})$. Thus we are in the case (vi).

For $(l, \mathcal{H}^{n+1}) = (4, 1)$, we have $\Delta(X, \mathcal{H}) = 0$. Therefore $(X, L) \cong (\mathbf{P}^{n+1}, \mathcal{O}_{\mathbf{P}^{n+1}}(4))$, which is the case (i).

The case of $h^0(A, \mathcal{H}_A) = n + 1$. Since $\mathcal{H}_A^n = 4$, we have $l \neq 1$, hence

(I) $\Delta(X, \mathcal{H}) = \mathcal{H}^{n+1} = 1$;

(II) $\Delta(X, \mathcal{H}) = \mathcal{H}^{n+1} = 2$.

Let $H_1, \ldots, H_n \in |\mathcal{H}|$ be general members, and put $X_k := \bigcap_{k\le i\le n} H_i$ for every $1 \le k \le n$. Then each X_k is a k-dimensional submanifold of X due to [6, (13.1)] and [5, (4.1)]. Moreover, by combining $H^1(X, \mathcal{O}_X) = 0$ and the Lefschetz-type theorem [7, (7.1.4)], we see that the ladder $\{X_k\}_{1\le k\le n+1}$ is regular, where we put $X_{n+1} := X$. Therefore we have $h^0(X_k, \mathcal{H}_{X_k}) = k$ for all $1 \le k \le n+1$. Since L_{X_1} is very ample and has degree 4, we have $g(X, \mathcal{H}) = g(X_1) = 1$ or 3. Then we argue case by case.

For the case $g(X, \mathcal{H}) = 1$, we are in the case (I) by [7, (12.3)] and $\mathrm{Pic}(X) \cong \mathbf{Z}$. Hence $(X, \mathcal{H})$ is a Del Pezzo manifold of degree one, which is the case (v) in the Theorem.

For the case $g(X, \mathcal{H}) = 3$ and (I), we are in the case (vii) from the following

Proposition 3.1. *Assume that* $g(X, \mathcal{H}) = 3$ *and* (I). *Then* $(X, \mathcal{H}) \cong (W_{12}, \mathcal{O}_{W_{12}}(1))$, *where* $W_{12} \subset \mathbf{P}(4, 3, 1^{n+1})$ *is a smooth weighted hypersurface of degree* 12.

Proof. We first note that X_1 is isomorphic to a plane quartic curve because of $g(X_1) = 3$. Next, we will show that

(1) $R(X_1, \mathcal{H}_{X_1}) \cong \mathbf{C}[x, y, z]/(F_{12})$, where $\mathrm{wt}(x, y, z) = (4, 3, 1)$ and $F_{12} = x^3 + y^4 + z\psi_{11}$ for some homogeneous polynomial $\psi_{11} \in \mathbf{C}[x, y, z]$ of degree 11; and
(2) The restriction map $\rho\colon R(X_2, \mathcal{H}_{X_2}) \to R(X_1, \mathcal{H}_{X_1})$ is surjective.

It suffices to prove the above: In fact, from (1) and (2), we see that X_2 is a weighted hypersurface of degree 12 in $\mathbf{P}(4, 3, 1^2)$, and therefore the assertion follows from [14, Proposition 3.10].

(1) We find the generators of $R(X_1, \mathcal{H}_{X_1})$ and the relations among them by using the Riemann-Roch theorem for X_1. By the sectional genus formula, we obtain $K_{X_1} = 4\mathcal{H}_{X_1}$. Therefore we have

$$h^0(l\mathcal{H}_{X_1}) = h^0((4 - l)\mathcal{H}_{X_1}) + l - 2.$$

For all $l \ge 5$, we see $h^0(l\mathcal{H}_{X_1}) = l - 2$. For $l \le 4$, we get the following table because of the well-known fact that a smooth plane quartic has no g^1_2.

Let z be a basis of the vector space $H^0(\mathcal{H}_{X_1})$. Choose $y \in H^0(3\mathcal{H}_{X_1})$ such that $H^0(3\mathcal{H}_{X_1}) = \langle y, z^3\rangle$. Similarly, choose $x \in H^0(4\mathcal{H}_{X_1})$ such that $H^0(4\mathcal{H}_{X_1}) = \langle x, yz, z^4\rangle$. From now on, we proceed in two steps.

Table

l	$h^0(l\mathcal{H}_{X_1})$	l	$h^0(l\mathcal{H}_{X_1})$
1	1	3	2
2	1	4	3

Step 1. We claim that the graded ring $R(X_1, \mathcal{H}_{X_1})$ is generated by three elements x, y, z. Indeed, it suffices to show that there exist some monomials in x, y, z which form a basis of $H^0(l\mathcal{H}_{X_1})$ for each $l \ge 5$.

We use induction on l. By the assumption (I), we see that $\mathrm{Bs}\,|\mathcal{H}|$ is a single point p. Note that each monomial in x, y contained in $H^0(l\mathcal{H}_{X_1})$ has a pole of order exactly l at p. When $l = 5$, we see that the monomials xz, yz^2, z^5 are linearly independent by comparing their orders of poles at p, hence form a basis of $H^0(5\mathcal{H}_{X_1})$.

Suppose that the assertion holds for $l - 1 \ge 5$. Note that $h^0(l\mathcal{H}_{X_1}) = h^0((l-1)\mathcal{H}_{X_1}) + 1$. It is easily shown that

> for two coprime positive integers a, b and an integer l with $l \ge (a-1)(b-1)$, the equation $ai + bj = l$ has at least one solution (i, j) of non-negative integers.

Set $(a, b) = (4, 3)$. Then, due to $l \ge 6$, there exists at least one section written as $x^i y^j$ $(i, j \ge 0)$ in $H^0(l\mathcal{H}_{X_1})$, not contained in $zH^0((l-1)\mathcal{H}_{X_1})$. Hence $H^0(l\mathcal{H}_{X_1}) = \mathbf{C}x^i y^j \oplus zH^0((l-1)\mathcal{H}_{X_1})$. From the induction hypothesis, the assertion holds for l. This proves our claim.

By Step 1, there exists a surjective homomorphism of graded rings

$$\Phi\colon \mathbf{C}[x, y, z] \to R(X_1, \mathcal{H}_{X_1}).$$

Step 2. We show that there exists an irreducible homogeneous polynomial F_{12} of degree 12 in $\mathbf{C}[x, y, z]$ such that $\mathrm{Ker}(\Phi) = (F_{12})$. Indeed, there exist no relations of degree $l < 12$ since the equation $4i + 3j = l$ has at most one solution (i, j) of non-negative integers. For $l = 12$, there are exactly 11 monomials in x, y, z of degree 12. On the other hand, $h^0(12\mathcal{H}_{X_1}) = 10$. Therefore there exists one relation F_{12} of degree 12, which is written as

$$F_{12} = x^3 + y^4 + z\psi_{11}(x, y, z)$$

after we replace x and y by suitable scalar multiples, where ψ_{11} is a homogeneous polynomial in x, y, z of degree 11.

It turns out that F_{12} is irreducible as follows. We can show that $x^3 + y^4$ is irreducible, immediately. Write $F_{12} = P_1(x, y, z)P_2(x, y, z)$ with some $P_1, P_2 \in \mathbf{C}[x, y, z]$. Without loss of generality, we may assume

$P_1(x,y,0)=1$. Hence $P_1(x,y,z)=1+z\xi_1$ and $P_2=x^3+y^4+z\xi_2$, where ξ_1,ξ_2 are polynomials in x,y,z. We have

$$\psi_{11}(x,y,z)=\xi_1(x^3+y^4+z\xi_2)+\xi_2.$$

It follows that $\xi_1=0$. Indeed, otherwise, the highest term of the right-hand side has degree ≥ 12, which is absurd. Therefore F_{12} is irreducible.

Moreover, combining this and the fact that $\mathrm{ht}(\mathrm{Ker}(\Phi))\leq\dim\mathbf{C}[x,y,z]-\dim R(X_1,\mathcal{H}_{X_1})=1$, we obtain $\mathrm{Ker}(\Phi)=(F_{12})$. Thus (1) is proved.

(2) It suffices to prove that $R(X_2,\mathcal{H}_{X_2})$ is Cohen-Macaulay, which is equivalent to finding a regular sequence of length $\dim R(X_2,\mathcal{H}_2)=3$ contained in $R(X_2,\mathcal{H}_{X_2})_+:=\bigoplus_{l>0}H^0(X_2,l\mathcal{H}_{X_2})$.

Before proving this, we fix our notation: Let $\mathbf{s}=\{s_0,\dots,s_N\}$ be a minimal set of generators of $R(X_2,\mathcal{H}_{X_2})$. Then there exists an isomorphism

$$R(X_2,\mathcal{H}_{X_2})\cong\mathbf{C}[s_0,\dots,s_N]/I_{\mathbf{s}},$$

where $I_{\mathbf{s}}$ is the homogeneous ideal defining X_2.

First we find a regular sequence of length 2 contained in $R(X_2,\mathcal{H}_{X_2})_+$ as follows: Since $h^0(X_2,\mathcal{H}_{X_2})=2$, we see that $H^0(\mathcal{H}_{X_2})$ has a basis $\{s,t\}$ satisfying

$$\rho(s)=z \text{ and } (t)_0=X_1.$$

We may assume that $\mathbf{s}$ contains these two elements. It is easy to check that $t,s\in R(X_2,\mathcal{H}_{X_2})_+$ form a regular sequence of length 2.

Next, we find an $R(X_2,\mathcal{H}_{X_2})/(t,s)$-regular element. One needs some information about generators of $I_{\mathbf{s}}$. For each $l\geq 0$, let

$$\rho_l\colon H^0(l\mathcal{H}_{X_2})\twoheadrightarrow H^0(l\mathcal{H}_{X_2})/\langle t\rangle\hookrightarrow H^0(l\mathcal{H}_{X_1}).$$

denote the restriction map. We proceed in two steps.

Step 1. We show that the ideal $I_{\mathbf{s}}$ has no generators in degrees ≤ 4 as follows: Firstly, we see that

$$\text{(A)}\qquad \mathrm{Im}(\rho_4)=H^0(4\mathcal{H}_{X_2})$$

combining $h^0(4\mathcal{H}_{X_1})=3$, the very ampleness of $L=4\mathcal{H}$ and the irrationality of X_1.

Subsequently, we find a basis of $H^0(l\mathcal{H}_{X_2})$ for each $1\leq l\leq 4$.

For $l=1$, there exist no relations in $H^0(\mathcal{H}_{X_2})$ by virtue of the minimality of $\mathbf{s}$.

For $l=2$, there are no relations: In fact, it follows that $H^0(2\mathcal{H}_{X_2})=\langle s^2,st,t^2\rangle$. Indeed, for any $\eta\in H^0(2\mathcal{H}_{X_2})$, we can write $\rho_2(\eta)=cz^2$ with some $c\in\mathbf{C}$. Therefore we see that η is a linear combination of s^2,st,t^2. These three monomials are linearly independent because each order of pole along X_1 differs from that of the others.

For $l=3$, we note that $1\leq\mathrm{rank}(\rho_3)\leq h^0(3\mathcal{H}_{X_1})=2$. We argue whether there are relations or not, case by case. We first suppose $\mathrm{rank}(\rho_3)=1$. Then, by the same argument as in the case $l=2$, we see $H^0(3\mathcal{H}_{X_2})=\langle s^3,s^2t,st^2,t^3\rangle$, which asserts that there are no relations. By (A), there exist sections $u,v\in H^0(4\mathcal{H}_{X_2})$ such that $\rho_4(u)=x,\rho_4(v)=yz$. It is easy to see that $H^0(4\mathcal{H}_{X_2})=\langle u,v,s^4,s^3t,s^2t^2,st^3,t^4\rangle$, therefore there are no relations in $H^0(4\mathcal{H}_{X_2})$.

Next, suppose that $\mathrm{rank}(\rho_3)=2$. Let w denote a section such that $\rho_3(w)=y$. Then we see that

$$H^0(3\mathcal{H}_{X_2})=\langle w,s^3,s^2t,st^2,t^3\rangle,$$
$$H^0(4\mathcal{H}_{X_2})=\langle u,sw,tw,s^4,s^3t,s^2t^2,st^3,t^4\rangle,$$

where u is a section such that $\rho_4(u)=x$. Therefore there exist no relations. In this way, it turns out that $I_{\mathbf{s}}$ has no generators in degrees ≤ 4.

Step 2. We claim that there exists an $R(X_2,\mathcal{H}_{X_2})/(t,s)$-regular element. Let u denote a section of $H^0(4\mathcal{H}_{X_2})$ such that $\rho_4(u)=x$. We assert that u is $R(X_2,\mathcal{H}_{X_2})/(t,s)$-regular. Indeed, $\mathrm{Proj}(R(X_2,\mathcal{H}_{X_2})/(t,s))$ is an integral scheme p because of $\mathcal{H}_{X_2}^2=1$. Thus we see that $(R(X_2,\mathcal{H}_{X_2})/(t,s))_+$ has no zero-divisors. Let m be a homogeneous element of degree a in $R(X_2,\mathcal{H}_{X_2})/(t,s)$ such that $um=0$. If $a>0$, we have $m=0$ obviously. If $a=0$, then we obtain $a=0$ by Step 1. Therefore our claim is proved.

Consequently, due to (1) and (2), the proposition is proved. □

For the case $g(X,\mathcal{H})=3$ and (II), we have $K_X=(2-n)\mathcal{H}$. Hence it follows that $H^1(X_3,m\mathcal{H}_{X_3})=0$ for all $m\geq 0$. We also see that the restriction map

$$\text{(B)}\qquad \varrho_m\colon H^0(X_2,m\mathcal{H}_{X_2})\to H^0(X_1,m\mathcal{H}_{X_1})$$

is surjective for all $m\geq 0$.

Proposition 3.2. *Assume that* $g(X,\mathcal{H})=3$ *and* (II). *Then* $L=2\mathcal{H}$ *is not very ample.*

Proof. Using (B), we obtain that $h^0(X_2,2\mathcal{H}_{X_2})=h^0(X_1,2\mathcal{H}_{X_1})+2=5$. Suppose that L is very ample. Then we see that L_{X_2} gives an embedding of X_2 into $\mathbf{P}^4$. But the double point formula for surfaces (see [2, Lemma 8.2.1]) $L_{X_2}^2(L_{X_2}^2-5)-10(g(X_2,L_{X_2})-1)+12\chi(\mathcal{O}_{X_2})-$

$2K_{X_2}^2 = 0$ implies that $-7 + 3p_g(X_2) = 0$, which is absurd. □

Therefore we see that this case cannot occur, which completes the proof of the Theorem.

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