On holomorphic curves extremal for the truncated defect relation

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Abstract: We consider a holomorphic curve from the complex plane into the complex projective space of odd dimension and give some results on truncated defects when the truncated defect relation is extremal.

Key words: Holomorphic curve; truncated defect relation; extremal.

1. Introduction. Let $f = [f_1, \dots, f_{n+1}]$ be a transcendental holomorphic curve from $\mathbf{C}$ into the n-dimensional complex projective space $P^n(\mathbf{C})$ with a reduced representation $(f_1, \dots, f_{n+1})\colon \mathbf{C} \to \mathbf{C}^{n+1} - \{\mathbf{0}\}$, where n is a positive integer. We suppose throughout the paper that f is linearly non-degenerate over $\mathbf{C}$; namely, $f_1, \dots, f_{n+1}$ are linearly independent over $\mathbf{C}$. For a vector $\boldsymbol{a} = (a_1, \dots, a_{n+1}) \in \mathbf{C}^{n+1} - \{\mathbf{0}\}$, let $\delta(\boldsymbol{a}, f)$ and $\delta_n(\boldsymbol{a}, f)$ be the deficiency and the truncated deficiency of $\boldsymbol{a}$ with respect to f respectively (see [7, Introduction]). We have that $0 \le \delta(\boldsymbol{a}, f) \le \delta_n(\boldsymbol{a}, f) \le 1$. Let X be a subset of $\mathbf{C}^{n+1} - \{\mathbf{0}\}$ in N-subgeneral position such that $\#X \ge N + 1$, where N is an integer satisfying $N \ge n$.

Cartan ([1], $N = n$) and Nochka ([4], $N > n$) gave the following

Theorem A (the truncated defect relation) (see [2, Corollary 3.3.9]). For any q elements $\boldsymbol{a}_j$ $(j = 1, \dots, q)$ of X $(2N - n + 1 \le q \le \infty)$, we have the inequality:

$$\sum_{j=1}^{q} \delta_n(\boldsymbol{a}_j, f) \le 2N - n + 1.$$

We are interested in the holomorphic curve f extremal for the truncated defect relation:

$$\sum_{j=1}^{q} \delta_n(\boldsymbol{a}_j, f) = 2N - n + 1. \tag{1}$$

In [6, Theorems 5.1, 6.1] we proved the following theorem when n is even:

Theorem B. Suppose that there are vectors $\boldsymbol{a}_1, \dots, \boldsymbol{a}_q$ of X such that (1) holds, where $2N - n + 1 < q \le \infty$. If $N > n = 2m$ $(m \in \mathbf{N})$, then $\#\{\boldsymbol{a}_j \mid \delta_n(\boldsymbol{a}_j, f) = 1\} > (2N - n + 1)/(n + 1)$.

In [8, Theorem 3.1] we proved a theorem for the holomorphic curve f with maximal deficiency sum with respect to $\delta(\boldsymbol{a}, f)$ when n is odd and $q < \infty$.

The purpose of this paper is to give a result when $N > n$, n is odd and (1) holds, which is an improvement of [8, Theorem 3.1].

2. Preliminaries and lemma. Let f, X etc. be as in Section 1, q an integer satisfying $2N - n + 1 \le q < \infty$ and we put $Q = \{1, 2, \dots, q\}$. Let $\{\boldsymbol{a}_j \mid j \in Q\}$ be a subset of X. For a non-empty subset P of Q, we denote by $V(P)$ the vector space spanned by $\{\boldsymbol{a}_j \mid j \in P\}$ and by $d(P)$ the dimension of $V(P)$. We put $\mathcal{O} = \{P \subset Q \mid 0 < \#P \le N + 1\}$.

Lemma 2.1 (see [2, (2.4.3), p. 68]). *If* $P \in \mathcal{O}$, *then* $\#P - d(P) \le N - n$.

For $\{\boldsymbol{a}_j \mid j \in Q\}$, let $\omega\colon Q \to (0, 1]$ be the Nochka weight function and θ the reciprocal number of the Nochka constant given in [2, p. 72]. We need the following properties of them:

Lemma 2.2 (see [2, Theorem 2.11.4]).

(a) $0 < \omega(j)\theta \le 1$ *for all* $j \in Q$;

(b) *If* $P \in \mathcal{O}$, *then* $\sum_{j \in P} \omega(j) \le d(P)$.

Definition 2.1 ([5, Definition 1]). We put

$$\lambda = \min_{P \in \mathcal{O}} d(P)/\#P \quad \text{and} \quad \sigma(j) = \lambda \quad (j \in Q).$$

Then, λ and σ have the following properties.

Lemma 2.3 ([5, Proposition 2]).

(a) $1/(N - n + 1) \le \lambda \le (n + 1)/(N + 1)$;

(b) *For any* $P \in \mathcal{O}$, $\sum_{j \in P} \sigma(j) \le d(P)$.

Remark 2.1.

(a) If $\lambda < (n + 1)/(2N - n + 1)$, then $\lambda = \min_{1 \le j \le q} \omega(j)$, $\omega(j) = \lambda$ and $\theta\omega(j) < 1$

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$(j \in P_0)$ for an element $P_0 \in \mathcal{O}$ satisfying $\lambda = d(P_0)/\#P_0$.

(b) If $\lambda \geq (n+1)/(2N-n+1)$, then $\omega(j) = 1/\theta = (n+1)/(2N-n+1)$ $(j = 1, \ldots, q)$.

(See the proof of [2, Proposition 2.4.4, p. 68] and the definitions of $\omega(j)$ and θ ([2, p. 72]).)

We introduce the following class of mappings from Q to $(0, 1]$:

Definition 2.2. $\mathcal{W} = \{\tau \colon Q \to (0,1] \mid \forall P \in \mathcal{O}, \sum_{j\in P} \tau(j) \leq d(P)\}$.

For example the Nochka weight function ω (by Lemma 2.2 (b)) and σ given in Definition 2.1 (by Lemma 2.3 (b)) are in $\mathcal{W}$.

Lemma 2.4. *For any $\tau \in \mathcal{W}$ it holds that*

(a) ([6, Lemma 2.9]) $\sum_{j=1}^{q} \tau(j)\delta_n(\boldsymbol{a}_j, f) \leq n+1$.

In particular,

(b) ([2, Th. 3.3.8]) $\sum_{j=1}^{q} \omega(j)\delta_n(\boldsymbol{a}_j, f) \leq n+1$.

Lemma 2.5 ([6, Corollary 2.2]). *Suppose that $N > n$ and that for $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_q \in X$, the equality* (1) *holds. For $j \in Q$ if $\theta\omega(j) < 1$, then $\delta_n(\boldsymbol{a}_j, f) = 1$.*

Corollary 2.1. *Suppose that $N > n \geq 2$ and that for $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_q \in X$ $(q < \infty)$, the equality* (1) *holds. If the inequality* $(*)$ $\lambda < (n+1)/(2N-n+1)$ *holds, then there exists a non-empty subset $P_0 \in \mathcal{O}$ satisfying*

(a) $d(P_0)/\#P_0 < (n+1)/(2N-n+1)$;

(b) $\delta_n(\boldsymbol{a}_j, f) = 1$ $(j \in P_0)$.

In particular,

$$\#\{j \in Q \mid \delta_n(\boldsymbol{a}_j, f) = 1\} > (2N-n+1)/(n+1).$$

Proof. By the definition of λ and the inequality $(*)$, there is a set $P_0 \in \mathcal{O}$ such that

$$d(P_0)/\#P_0 = \lambda < (n+1)/(2N-n+1).$$

By $(*)$ and Remark 2.1 (a), we have $\omega(j) = \lambda < \theta^{-1}$ $(j \in P_0)$, so that $\theta\omega(j) < 1$ $(j \in P_0)$. By Lemma 2.5, $\delta_n(\boldsymbol{a}_j, f) = 1$ $(j \in P_0)$ since (1) is assumed. As

$$\#P_0 = d(P_0)/\lambda > \frac{2N-n+1}{n+1} d(P_0) \geq \frac{2N-n+1}{n+1},$$

we have our corollary. □

Let $\mathcal{F}$ be a family of non-empty subsets of X.

Definition 2.3 ([8, Definition 2.2]). We say that two sets $P_1, P_2 \in \mathcal{F}$ have a relation $P_1 \sim P_2$ if and only if either (i) $P_1 \cap P_2 \neq \emptyset$ or (ii) there exist sets $R_1, \ldots, R_s \in \mathcal{F}$ such that

$$R_{j-1} \cap R_j \neq \emptyset \quad (1 \leq j \leq s+1),\ R_0 = P_1,\ R_{s+1} = P_2.$$

Lemma 2.6 ([8, Lemma 2.8]). *The relation "$\sim$" in $\mathcal{F}$ is an equivalence relation.*

Proof. As the proof is not given in [8], we give it here.

(i) The relation "$\sim$" is reflexive. It is trivial that for any $P \in \mathcal{F}$, $P \sim P$.

(ii) The relation "$\sim$" is symmetric. We prove that for $P_1, P_2 \in \mathcal{F}$, if $P_1 \sim P_2$, then $P_2 \sim P_1$.

Case 1: $P_1 \cap P_2 \neq \emptyset$. Then, $P_2 \cap P_1 \neq \emptyset$ and we have $P_2 \sim P_1$.

Case 2: There exist sets $R_1, \ldots, R_s \in \mathcal{F}$ such that $R_{j-1} \cap R_j \neq \emptyset$ $(1 \leq j \leq s+1)$, where $R_0 = P_1$ and $R_{s+1} = P_2$. Put $R_{s+1-j} = T_j$ $(0 \leq j \leq s+1)$. Then, $T_1, \ldots, T_s \in \mathcal{F}$, $T_{j-1} \cap T_j \neq \emptyset$ $(1 \leq j \leq s+1)$, $T_0 = P_2$ and $T_{s+1} = P_1$. This means that $P_2 \sim P_1$.

(iii) The relation "$\sim$" is transitive. We prove that for $P_1, P_2, P_3 \in \mathcal{F}$, if $P_1 \sim P_2$ and $P_2 \sim P_3$ then $P_1 \sim P_3$.

Case 1: $P_1 \cap P_2 \neq \emptyset$ and $P_2 \cap P_3 \neq \emptyset$. We put $R_1 = P_2$. then R_1 satisfies the condition (ii) of Definition 2.3 and so $P_1 \sim P_3$.

Case 2: $P_1 \cap P_2 \neq \emptyset$ and there exist sets $T_1, \ldots, T_t \in \mathcal{F}$ such that $T_{j-1} \cap T_j \neq \emptyset$ $(1 \leq j \leq t+1)$, where $T_0 = P_2$ and $T_{t+1} = P_3$. In this case, we put

$$R_0 = P_1,\ R_1 = P_2,\ R_{j+1} = T_j \quad (1 \leq j \leq t+1).$$

Then, the sets $R_0, R_1, \ldots, R_{t+2}$ satisfy the condition (ii) of Definition 2.3 and so $P_1 \sim P_3$.

Case 3: There exist sets $S_1, \ldots, S_s \in \mathcal{F}$ such that $S_{j-1} \cap S_j \neq \emptyset$ $(1 \leq j \leq s+1)$, where $S_0 = P_1$, $S_{s+1} = P_2$ and $P_2 \cap P_3 \neq \emptyset$. In this case we have $P_1 \sim P_3$ as in Case 2.

Case 4: There exist sets $S_1, \ldots, S_s \in \mathcal{F}$ such that $S_{j-1} \cap S_j \neq \emptyset$ $(1 \leq j \leq s+1)$, where $S_0 = P_1, S_{s+1} = P_2$ and there exist sets $T_1, \ldots, T_t \in \mathcal{F}$ such that $T_{j-1} \cap T_j \neq \emptyset$ $(1 \leq j \leq t+1)$, where $T_0 = P_2$ and $T_{t+1} = P_3$. In this case, we put $R_0 = P_1$, $R_j = S_j$ $(1 \leq j \leq s)$, $R_{s+1} = P_2$, $R_{s+1+j} = T_j$ $(1 \leq j \leq t)$, $R_{s+t+2} = P_3$. Then the sets $R_0, R_1, \ldots, R_{s+t+2}$ satisfy the condition (ii) of Definition 2.3 and so $P_1 \sim P_3$. □

3. Extremal case I: $q < \infty$.

Let f, X, $\delta_n(\boldsymbol{a}, f)$, $\mathcal{O}$ etc. be as in Section 1 or 2. The purpose of this section is to give a result when n is odd and the trucated defect relation is extremal for $q = \#\{\boldsymbol{a} \in X \mid \delta_n(\boldsymbol{a}, f) > 0\} < \infty$. We put

$$\{\boldsymbol{a} \in X \mid \delta_n(\boldsymbol{a}, f) > 0\} = \{\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_q\}.$$

We suppose that

(3.i) $N > n = 2m-1$ $(m \in \mathbf{N})$;

(3.ii) $\sum_{j=1}^{q} \delta_n(\boldsymbol{a}_j, f) = 2N - n + 1$.

From (3.ii), the number q must satisfy the inequality $2N-n+1 \leq q < \infty$. We can apply lemmas in Section 2. We note that $(n+1)/(2N-n+1) = m/(N-m+1)$ as $n = 2m-1$.

From Lemma 2.3 (b), Lemma 2.4 (a) and the assumption (3.ii), we obtain the inequality $\lambda \leq m/(N-m+1)$.

First, we have the following

Lemma 3.1. *If* $\lambda < m/(N-m+1)$, *then there exists* $P_0 \in \mathcal{O}$ *satisfying* $\delta_n(\boldsymbol{a}_j, f) = 1$ $(j \in P_0)$ *and*

$$\#P_0 = d(P_0)/\lambda > \frac{2N-n+1}{n+1} d(P_0) \geq \frac{2N-n+1}{n+1}.$$

Proof. By Lemma 2.3 (a) we have $m \geq 2$, so that $n = 2m-1 \geq 3$. We can apply Corollary 2.1 to obtain this lemma. □

Next, we consider the case when $\lambda = m/(N-m+1)$. We note that $\omega(j) = \lambda$ $(j \in Q)$ by Remark 2.1 (b). Put

$$\mathcal{O}_1 = \{P \in \mathcal{O} \mid d(P)/\#P = \lambda = m/(N-m+1)\}.$$

Note that $\mathcal{O}_1$ is non-empty and finite. We apply Definition 2.3 and Lemma 2.6 to $\mathcal{F} = \mathcal{O}_1$ and classify $\mathcal{O}_1$ by the equivalence relation "$\sim$." We put

$$\mathcal{O}_1/\!\sim\, = \{\mathcal{P}_1, \dots, \mathcal{P}_p\};$$
$$M_k = \bigcup_{P \in \mathcal{P}_k} P \quad (k = 1, \dots, p).$$

The method used in [8, Section 3] is applicable to this case and we obtain the followings. As in [8, Proposition 3.5] we have the following

Lemma 3.2.
(a) $M_k \in \mathcal{O}_1$ $(1 \leq k \leq p)$;
(b) $p \geq 2$;
(c) $M_k \cap M_\ell = \emptyset$ $(k \neq \ell)$ *and*
(d) $d(M_k) = m$, $\#M_k = N-m+1$ $(1 \leq k \leq p)$.

Put $Q_o = \bigcup_{k=1}^p M_k$. As in [8, Proposition 3.6] we have the following

Lemma 3.3.
(a) $Q = Q_o$;
(b) $(N-m+1)|q$ *and* $p = q/(N-m+1)$.

As in [8, Proposition 3.7] we have the following

Lemma 3.4. *Any* m *elements of* $\{\boldsymbol{a}_1, \dots, \boldsymbol{a}_q\}$ *are linearly independent*.

Summarizing Lemmas 3.1, 3.2, 3.3 and 3.4 we obtain the following

Theorem 3.1. *Suppose that*
(i) $N > n = 2m-1$ $(m \in \mathbf{N})$;
(ii) $\delta_n(\boldsymbol{a}_j, f) > 0$ $(j = 1, \dots, q; q < \infty)$ *and*

$$\sum_{j=1}^{q} \delta_n(\boldsymbol{a}_j, f) = 2N-n+1.$$

Then, for the set $Q = \{1, \dots, q\}$, either (I) or (II) given below holds:
(I) $\#\{j \in Q \mid \delta_n(\boldsymbol{a}_j, f) = 1\} > \frac{2N-n+1}{n+1}$.
(II) q is divisible by $N-m+1$ and for $p = q/(N-m+1)$, there are mutually disjoint subsets $M_1, \dots, M_p$ of Q satisfying
  (a) $Q = \bigcup_{k=1}^p M_k$;
  (b) $d(M_k) = m$, $\#M_k = N-m+1$ $(1 \leqq k \leqq p)$;
  (c) any m elements of $\{\boldsymbol{a}_1, \dots, \boldsymbol{a}_q\}$ are linearly independent.

4. Extremal case II: $q = \infty$. Let f, X etc. be as in Section 1 or 2. As in the case of meromorphic functions (see [3, p. 79]), the set $Y = \{\boldsymbol{a} \in X \mid \delta_n(\boldsymbol{a}, f) > 0\}$ is at most countable. We treated the case when Y is a finite set in Section 3. In this section, we suppose that Y is not finite and we put $Y = \{\boldsymbol{a}_j \mid j \in \mathbf{N}\}$, where $\mathbf{N}$ is the set of positive integers. We put

$$\mathcal{O}_\infty = \{P \subset \mathbf{N} \mid 0 < \#P \leq N+1\}$$

and for any non-empty finite subset P of $\mathbf{N}$, we use $V(P)$ and $d(P)$ as in Section 2. We put $\mu = \min_{P \in \mathcal{O}_\infty} d(P)/\#P$. Note that the set $\{d(P)/\#P \mid P \in \mathcal{O}_\infty\}$ is a finite set. We have the following
(4.a) ([5, p. 144]) $\frac{1}{N-n+1} \leq \mu \leq \frac{n+1}{N+1}$;
(4.b) ([6, Lemma 4.1]) $\sum_{j=1}^\infty \delta_n(\boldsymbol{a}_j, f) \leq (n+1)/\mu$.

From now on throughout this section we suppose that
(4.i) $N > n = 2m-1$ $(m \in \mathbf{N})$;
(4.ii) $\sum_{j=1}^\infty \delta_n(\boldsymbol{a}_j, f) = 2N-n+1$.

From (4.ii) and (4.b), we have the following inequality:

$$\mu \leq (n+1)/(2N-n+1).$$

First, we have the following

Proposition 4.1. *If* $\mu < (n+1)/(2N-n+1)$, *then*

$$\#\{j \in \mathbf{N} \mid \delta_n(\boldsymbol{a}_j, f) = 1\} > (2N-n+1)/(n+1).$$

(For the proof of this proposition, see the latter half of the Proof of [6, Theorem 6.1, p. 17]. Note that $m \geq 2$ by (4.a).)

Next, we consider the case $\mu = (n+1)/(2N-n+1)$. Note that $\mu = (n+1)/(2N-n+1) = m/(N-m+1)$. We put

$$\mathcal{F}_0 = \left\{ P \in \mathcal{O}_\infty \;\middle|\; d(P)/\#P = \mu = \frac{m}{N-m+1} \right\},$$

which is not empty. Corresponding to [8, Propositions 3.2–3.7], we obtain the following propositions.

Proposition 4.2. *For any* $P \in \mathcal{F}_0$, $d(P) \leq m$ *and* $\#P \leq N - m + 1$.

Proof. Let P be in $\mathcal{F}_0$. Then, $\#P = d(P)/\mu$ and so we have the inequality

$$\#P - d(P) = d(P)(N-n)/m \leq N - n$$

by Lemma 2.1 and $n = 2m - 1$. This implies that $d(P) \leq m$ and $\#P \leq N - m + 1$. □

Proposition 4.3. *For any element* P_0 *of* $\mathcal{F}_0$, $\{P \in \mathcal{F}_0 \mid P - P_0 \neq \emptyset\} \neq \emptyset$.

Proof. Let P_0 be an element of $\mathcal{F}_0$ and put

$$\mathcal{F}_1 = \{P \in \mathcal{O}_\infty \mid P - P_0 \neq \emptyset\}.$$

Then, $\mathcal{F}_1 \neq \emptyset$ since $\#P_0 \leqq N - m + 1 < \infty$. As the set $\{d(P)/\#P \mid P \in \mathcal{F}_1\}$ is finite, we put

$$\mu_1 = \min_{P \in \mathcal{F}_1} d(P)/\#P.$$

Then, we have that $\mu = \mu_1$. In fact, the inequality $\mu \leq \mu_1$ holds by the definition of μ. Suppose that $\mu < \mu_1$ and let ϵ be any number satisfying

$$(2) \qquad 0 < \epsilon < 1 - \mu/\mu_1$$

and $P_1 \in \mathcal{F}_1$ satisfying $d(P_1)/\#P_1 = \mu_1$. We choose a positive integer q satisfying

(4.c) $P_0 \cup P_1 \subset Q = \{1, 2, \dots, q\}$;

(4.d) $\sum_{j=1}^{q} \delta_n(\boldsymbol{a}_j, f) > 2N - n + 1 - \epsilon$

and $2N - n + 1 < q < \infty$. For this Q, we use θ_q, ω_q and λ_q instead of θ, ω and λ in Section 2 respectively. By the choice of q in (4.c), $\mu = \lambda_q$ and by Remark 2.1 (b) for $j \in Q$

$$(3) \qquad \omega_q(j) = \mu = m/(N - m + 1)$$

and so we have from (4.d)

$$(4) \qquad \sum_{j=1}^{q} \omega_q(j)\delta_n(\boldsymbol{a}_j, f) > n + 1 - \epsilon\mu.$$

Put

$$\tau(j) = \begin{cases} \mu & (j \in P_0) \\ \mu_1 & (j \in Q - P_0). \end{cases}$$

Then, the function $\tau \colon Q \to (0, 1]$ belongs to $\mathcal{W}$. In fact, for any element P of $\mathcal{O}_\infty$ such that $P \subset Q$,

(a) when $P \subset P_0$,

$$\sum_{j \in P} \tau(j) = \mu\#P \leq (d(P)/\#P)\#P = d(P);$$

(b) when $P - P_0 \neq \emptyset$,

$$\sum_{j \in P} \tau(j) \leq \mu_1\#P \leq (d(P)/\#P)\#P = d(P).$$

By Lemma 2.4 (a), (3) and (4) we obtain the inequality

$$\sum_{j=1}^{q} \tau(j)\delta_n(\boldsymbol{a}_j, f) \leq n + 1 < \sum_{j=1}^{q} \mu\delta_n(\boldsymbol{a}_j, f) + \epsilon\mu,$$

which reduces to the inequality

$$(5) \qquad (\mu_1 - \mu) \sum_{j \in Q - P_0} \delta_n(\boldsymbol{a}_j, f) < \epsilon\mu.$$

As

$$\sum_{j \in Q - P_0} \delta_n(\boldsymbol{a}_j, f) > 2N - n + 1 - \epsilon - \#P_0$$
$$\geq N - m + 1 - \epsilon,$$

from (5) we have the inequality

$$(\mu_1 - \mu)(N - m + 1 - \epsilon) < \epsilon\mu,$$

which reduces to the inequality

$$(1 - \mu/\mu_1)(N - m + 1) < \epsilon,$$

which contradicts (2) as $N - m \geq 1$. This implies that the equality $\mu = \mu_1$ must hold and P_1 belongs to $\mathcal{F}_0$ and satisfies that $P_1 - P_0 \neq \emptyset$. □

Proposition 4.4. *Let* P_1 *and* P_2 *be in* $\mathcal{F}_0$. *If* $P_1 \cap P_2 \neq \emptyset$, *then* $P_1 \cup P_2 \in \mathcal{F}_0$.

Proof. As $P_1, P_2 \in \mathcal{F}_0$,

$$(6) \qquad d(P_1)/\#P_1 = d(P_2)/\#P_2 = \mu.$$

From Proposition 4.2 we obtain the inequality

$$(7) \qquad d(P_1) + d(P_2) \leq 2m = n + 1.$$

As

$$(8) \qquad d(P_1 \cup P_2) + d(P_1 \cap P_2) \leq d(P_1) + d(P_2)$$

(see [2, p. 68]) and $d(P_1 \cap P_2) \geq 1$ by our assumption, from (7) and (8) we obtain the inequality

$$d(P_1 \cup P_2) \leq n,$$

which implies that $\#(P_1 \cup P_2) \leq N$ so that $P_1 \cup P_2 \in \mathcal{O}_\infty$.

Next, by the definition of μ, we have the inequalities

$$(9) \qquad \mu \leq \frac{d(P_1 \cup P_2)}{\#(P_1 \cup P_2)} \quad \text{and} \quad \mu \leq \frac{d(P_1 \cap P_2)}{\#(P_1 \cap P_2)}.$$

We note that $P_1 \cap P_2 \in \mathcal{O}_\infty$ since $0 < \#(P_1 \cap P_2) \leq N - m + 1 \leq N$.

From (6), (8) and (9) we have the inequality

$$\mu \le \frac{d(P_1 \cup P_2)}{\#(P_1 \cup P_2)} \le \frac{d(P_1) + d(P_2) - d(P_1 \cap P_2)}{\#P_1 + \#P_2 - \#(P_1 \cap P_2)} \le \mu,$$

which implies that $d(P_1 \cup P_2)/\#(P_1 \cup P_2) = \mu$, so that $P_1 \cup P_2 \in \mathcal{F}_0$. □

We apply Definition 2.3 and Lemma 2.6 to $\mathcal{F} = \mathcal{F}_0$ and classify $\mathcal{F}_0$ by the equivalence relation "$\sim$." We put

$$\mathcal{F}_0/\!\sim\, = \{\mathcal{P}_1, \dots, \mathcal{P}_p\} \quad (1 \le p \le \infty);$$
$$M_k = \bigcup_{P \in \mathcal{P}_k} P \quad (k = 1, \dots, p).$$

Corresponding to Lemma 3.2, we have the following

Proposition 4.5.
(a) $M_k \in \mathcal{F}_0$ $(1 \le k \le p)$;
(b) $p \ge 2$;
(c) $M_k \cap M_\ell = \emptyset$ $(k \ne \ell)$ *and*
(d) $d(M_k) = m$, $\#M_k = N - m + 1$ $(1 \le k \le p)$.

Proof. (a) First, we note that $\#\mathcal{P}_k \le N - m + 1$ by Propositions 4.2 and 4.4. By the definition of the relation "$\sim$" and by Proposition 4.4, we have this assertion.

(b) As M_1 belongs to $\mathcal{F}_0$, we apply Proposition 4.3 to M_1. There exists an element $P \in \mathcal{F}_0$ such that $P - M_1 \ne \emptyset$. In this case, $P \cap M_1 = \emptyset$. In fact, if $P \cap M_1 \ne \emptyset$, then, by the definition of the relation "$\sim$," $P \sim M_1$. This means that $P \in \mathcal{P}_1$, and so $P \subset M_1$ by the definition of M_1, which implies that $P - M_1 = \emptyset$. This is a contradiction. We have that $p \ge 2$.

(c) This is trivial from the definition of M_k.

(d) Suppose to the contrary that there exists at least one k $(1 \le k \le p)$ such that $d(M_k) \le m - 1$. For simplicity, we may suppose without loss of generality that $k = 1$. Then, as

$$d(M_1 \cup M_2) + d(M_1 \cap M_2) \le d(M_1) + d(M_2)$$

(see [2, p. 68]), by Proposition 4.2 and (a) of this proposition we have

$$d(M_1) + d(M_2) \le m - 1 + m = 2m - 1 = n,$$

which means that $M_1 \cup M_2 \in \mathcal{O}_\infty$. As $M_1, M_2 \in \mathcal{F}_0$, by the definition of μ we have

$$\mu \le \frac{d(M_1 \cup M_2)}{\#(M_1 \cup M_2)} \le \frac{d(M_1) + d(M_2)}{\#M_1 + \#M_2} = \mu.$$

Note that $M_1 \cap M_2 = \emptyset$ by (c) of this proposition. We have $d(M_1 \cup M_2)/\#(M_1 \cup M_2) = \mu$, which means that $M_1 \cup M_2 \in \mathcal{F}_0$. Then, as

$$M_1 \sim M_1 \cup M_2 \quad \text{and} \quad M_1 \cup M_2 \sim M_2,$$

we have that $M_1 \sim M_2$. This is a contradiction since $M_1 \in \mathcal{P}_1$ and $M_2 \in \mathcal{P}_2$. This implies that $d(M_k) = m$ and $\#M_k = N - m + 1$ $(k = 1, \dots, p)$. □

Put $\bigcup_{k=1}^{p} M_k = Q_o$. Then, we have the following

Proposition 4.6. $Q_o = \mathbf{N}$.

Proof. Suppose to the contrary that $Q_o \subsetneqq \mathbf{N}$. Put $\mathcal{F}_2 = \{P \in \mathcal{O}_\infty \mid P - Q_o \ne \emptyset\}$, which is not empty by our assumption of this proof, and we put $\mu_2 = \min_{P \in \mathcal{F}_2} d(P)/\#P$. Then, $\mu < \mu_2$. In fact, the inequality $\mu \le \mu_2$ holds in general by the definition of μ. Suppose that $\mu = \mu_2$. Then, there exists an element $P \in \mathcal{F}_2$ satisfying $d(P)/\#P = \mu_2 = \mu$, which means that $P \in \mathcal{F}_0$ and $P - Q_o \ne \emptyset$. This is a contradiction to the definition of Q_o. We have that $\mu < \mu_2$. Let $P_0 \in \mathcal{F}_0$ satisfying $d(P_0)/\#P_0 = \mu$, q_o the least number in $\mathbf{N} - Q_o$ and ϵ any number satisfying

$$(10) \qquad 0 < \epsilon < (\mu_2/\mu - 1)\delta_n(\boldsymbol{a}_{q_o}, f).$$

We choose a positive integer u satisfying
(4.e) $P_0 \subset Q = \{1, 2, \dots, u\}$ and $u > q_o$;
(4.f) $\sum_{j=1}^{u} \delta_n(\boldsymbol{a}_j, f) > 2N - n + 1 - \epsilon$

and $2N - n + 1 < u < \infty$. For this Q, we use θ_u, ω_u and λ_u instead of θ, ω and λ in Section 2 respectively. By the choice of u in (4.e), $\mu = \lambda_u$ and by Remark 2.1 (b) for $j \in Q$

$$(11) \qquad \omega_u(j) = \mu = m/(N - m + 1)$$

and so we have from (4.f)

$$(12) \qquad \sum_{j=1}^{u} \omega_u(j)\delta_n(\boldsymbol{a}_j, f) > n + 1 - \epsilon\mu.$$

Put

$$\tau(j) = \begin{cases} \mu & (j \in Q_o \cap Q) \\ \mu_2 & (j \in Q - Q_o). \end{cases}$$

Then, the function $\tau\colon Q \to (0, 1]$ belongs to $\mathcal{W}$ (see (a) and (b) in the Proof of Proposition 4.3). By Lemma 2.4 (a), (11) and (12) we obtain the inequality

$$\sum_{j=1}^{u} \tau(j)\delta_n(\boldsymbol{a}_j, f) \le n + 1 < \sum_{j=1}^{u} \mu\delta_n(\boldsymbol{a}_j, f) + \epsilon\mu,$$

which reduces to the inequality

$$(\mu_2 - \mu) \sum_{j \in Q - Q_o} \delta_n(\boldsymbol{a}_j, f) < \epsilon\mu,$$

so that we have the inequality

$$(\mu_2/\mu - 1)\delta_n(\boldsymbol{a}_{q_o}, f) < \epsilon,$$

which is a contradiction to (10). This means that $Q_o = \mathbf{N}$. □

Remark 4.1. p (= the number of elements of $\mathcal{F}_0/\sim) = \infty$.

In fact, if $p < \infty$, then by Propositions 4.5 (d) and 4.6, $\#\mathbf{N} = p(N - m + 1) < \infty$, which is a contradiction.

Proposition 4.7. *Any m elements of $\{\boldsymbol{a}_j \mid j \in \mathbf{N}\}$ are linearly independent.*

Proof. Let $\boldsymbol{b}_1, \dots, \boldsymbol{b}_m$ be any m vectors in $\{\boldsymbol{a}_j \mid j \in \mathbf{N}\}$. As $m < \infty$ there is a positive integer k such that $(*)$ $M_k \cap \{\boldsymbol{b}_1, \dots, \boldsymbol{b}_m\} = \emptyset$. We suppose without loss of generality that $k = 1$. As $d(M_1) = m$ by Propsition 4.5 (d), there are m linearly independent vectors $\boldsymbol{c}_1, \dots, \boldsymbol{c}_m$ in M_1. As $\#M_1 = N - m + 1$, $(*)$ implies that $\#(M_1 \cup \{\boldsymbol{b}_1, \dots, \boldsymbol{b}_m\}) = N + 1$. As X is in $N-$subgeneral position, there are $n+1 = 2m$ linearly independent vectors in $M_1 \cup \{\boldsymbol{b}_1, \dots, \boldsymbol{b}_m\}$. This implies that $n+1$ vectors $\boldsymbol{b}_1, \dots, \boldsymbol{b}_m, \boldsymbol{c}_1, \dots, \boldsymbol{c}_m$ are linearly independent since $d(M_1) = m$, and so $\boldsymbol{b}_1, \dots, \boldsymbol{b}_m$ are linearly independent. □

Summarizing Propositions 4.1, 4.5, 4.7 and Remark 4.1 we obtain the following

Theorem 4.1. *Suppose that*

(i) $N > n = 2m - 1$, *where* $m \in \mathbf{N}$;

(ii) *there exist an infinite number of vectors $\boldsymbol{a}_j$ in X satisfying $\delta_n(\boldsymbol{a}_j, f) > 0$ $(j \in \mathbf{N})$ and*

$$\sum_{j=1}^{\infty} \delta_n(\boldsymbol{a}_j, f) = 2N - n + 1.$$

Then, either (I) *or* (II) *given below holds*:

(I) $\#\{j \in \mathbf{N} \mid \delta_n(\boldsymbol{a}_j, f) = 1\} > \frac{2N-n+1}{n+1}$.

(II) *There are mutually disjoint subsets $M_1, M_2, \dots, M_k, \dots$ of $\mathbf{N}$ satisfying*

(a) $\mathbf{N} = \bigcup_{k=1}^{\infty} M_k$,

(b) $\#M_k = N - m + 1$, $d(M_k) = m$ $(k = 1, 2, \dots)$ *and*

(c) *any m elements of $\{\boldsymbol{a}_j \mid j \in \mathbf{N}\}$ are linearly independent.*

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