Classification of a family of Hamiltonian-stationary Lagrangian submanifolds in $\mathbf{C}^n$

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Abstract: A Lagrangian submanifold in the complex Euclidean n-space $\mathbf{C}^n$ is called Hamiltonian-stationary if it is a critical point of the area functional restricted to (compactly supported) Hamiltonian variations. In this article, we classify the family of Hamiltonian-stationary Lagrangian submanifolds of $\mathbf{C}^n$ which are Lagrangian H-umbilical.

Key words: Hamiltonian-stationary; H-umbilical submanifold; complex extensor.

1. Introduction. Let $\mathbf{C}^n$ be the complex Euclidean n-space with complex structure J and Kaehler metric $\langle\ ,\ \rangle$. The Kaehler 2-form ω is defined by $\omega(\cdot\,,\cdot) = \langle J\cdot,\cdot\rangle$. An immersion $\psi : M \to \mathbf{C}^n$ of an n-manifold M into $\mathbf{C}^n$ is called *Lagrangian* if $\psi^*\omega = 0$ on M. A vector field X on $\mathbf{C}^n$ is called Hamiltonian if $\mathcal{L}_X\omega = f\omega$ for some function $f \in C^\infty(\mathbf{C}^n)$, where $\mathcal{L}$ is the Lie derivative. Thus, there exists a smooth real-valued function φ on $\mathbf{C}^n$ such that $X = J\tilde{\nabla}\varphi$, where $\tilde{\nabla}$ is the gradient in $\mathbf{C}^n$. The diffeomorphisms of the the flux ψ_t of X satisfy $\psi_t\omega = e^{h_t}\omega$. Thus they transform Lagrangian submanifolds into Lagrangian submanifolds.

Oh [15] studied the following variational problem: A normal vector field ξ to a Lagrangian immersion $\psi : M^n \to \mathbf{C}^n$ is called Hamiltonian if $\xi = J\nabla f$, where f is a smooth function on M^n and ∇f is the gradient of f with respect to the induced metric.

If $f \in C_0^\infty(M)$ and $\psi_t : M \to \mathbf{C}^n$ is a variation of ψ with $\psi_0 = \psi$ and variational vector field ξ, then the first variation of the volume functional is

$$\frac{d}{dt}_{|t=0}\mathrm{vol}(M, \psi_t^* g) = -\int_M f\,\mathrm{div} JH dM,$$

where H is the mean curvature vector of the immersion ψ and div is the divergence operator on M. Critical points of this variational functional are called *Hamiltonian-stationary* (or Hamiltonian-minimal). Lagrangian submanifolds with parallel mean curvature vector are Hamiltonian-stationary.

Hamiltonian-stationary Lagrangian submanifolds in $\mathbf{C}^n$ (mostly in $\mathbf{C}^2$) have been studied in [1–7, 10, 12–15], among others.

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In this article, we classify the family of Hamiltonian-stationary Lagrangian submanifolds of $\mathbf{C}^n$ which are Lagrangian H-umbilical. A related result is also obtained.

2. Preliminaries. Let $f : M \to \mathbf{C}^n$ be an isometric immersion of a Riemannian n-manifold M into $\mathbf{C}^n$. We denote the Riemannian connections of M and $\mathbf{C}^n$ by ∇ and $\tilde{\nabla}$, respectively; and by D the connection on the normal bundle of the submanifold.

The formulas of Gauss and Weingarten are

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.1}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi \tag{2.2}$$

for tangent vector fields X, Y and normal vector field ξ. If we denote the Riemann curvature tensor of ∇ by R, then the equations of Gauss and Codazzi are given respectively by

$$\langle R(X,Y)Z, W\rangle = \langle h(X,W), h(Y,Z)\rangle - \langle h(X,Z), h(Y,W)\rangle\,, \tag{2.3}$$

$$(\nabla h)(X,Y,Z) = (\nabla h)(Y,X,Z), \tag{2.4}$$

where $(\nabla h)(X,Y,Z) = D_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$.

For a Lagrangian submanifold M of $\mathbf{C}^n$, we also have (cf. [11])

$$D_X JY = J\nabla_X Y, \tag{2.5}$$

$$\langle h(X,Y), JZ\rangle = \langle h(Y,Z), JX\rangle = \langle h(Z,X), JY\rangle\,. \tag{2.6}$$

We recall some definitions and results from [9].

By definition, a Lagrangian submanifold without totally geodesic points is called a *Lagrangian H-umbilical submanifold* if the second fundamental form takes the following simple form (cf. [9]):

$$(2.7)\quad h(e_1,e_1)=\lambda Je_1,\ h(e_j,e_j)=\mu, Je_1,\ j>1,\quad h(e_1,e_j)=\mu Je_j,\ h(e_j,e_k)=0,\ 2\le j\ne k\le n$$

for some functions λ, μ with respect to some suitable orthonormal local frame field $\{e_1,\dots,e_n\}$. Such submanifolds are known to be the simplest Lagrangian submanifolds next to the totally geodesic ones.

Let $G: N^{n-1}\to \mathbf{E}^n$ be an isometric immersion of a Riemannian $(n-1)$-manifold into the Euclidean n-space $\mathbf{E}^n$ and let $F: I\to \mathbf{C}^*$ be a unit speed curve in $\mathbf{C}^*=\mathbf{C}-\{0\}$. We may extend $G: N^{n-1}\to \mathbf{E}^n$ to an immersion of $I\times N^{n-1}$ into $\mathbf{C}^n$ as

$$(2.8)\qquad F\otimes G: I\times N^{n-1}\to \mathbf{C}\otimes\mathbf{E}^n=\mathbf{C}^n,$$

where $(F\otimes G)(s,p)=F(s)\otimes G(p)$ for $s\in I,\ p\in N^{n-1}$. This extension $F\otimes G$ of G via tensor product is called the *complex extensor* of G via F (or of the submanifold N^{n-1} via F).

Proposition 1. *Let $\iota: S^{n-1}\to\mathbf{E}^n$ be the inclusion of a hypersphere of $\mathbf{E}^m$ centered at the origin. Then every complex extensor $\phi=F\otimes\iota$ of ι via a unit speed curve $F: I\to\mathbf{C}^*$ is a Lagrangian H-umbilical submanifold of $\mathbf{C}^n$ unless F is contained in a line through the origin* (*which gives a totally geodesic Lagrangian submanifold*).

For $F\otimes\iota$, we choose e_1 a unit vector field tangent to the first factor and $e_2,\dots,e_n$ to the second factor of $I\times S^{n-1}$. Without loss of generality, we may assume ι is the inclusion $\iota_0^n: S^{n-1}(1)\subset\mathbf{E}^n$ of the unit hypersphere centered at the origin of $\mathbf{E}^n$.

If we put $F'=e^{i\varphi(s)}$ and $F=r(s)e^{i\theta(s)}$, then the second fundamental form of the complex extensor $F\otimes\iota_0^n$ satisfies (2.7) with

$$(2.9)\qquad \lambda=\varphi'(s)=\kappa,\quad \mu=\frac{\langle F', iF\rangle}{\langle F,F\rangle}=\theta'(s).$$

From (2.9) and Proposition 1 we see that a complex extensor is totally geodesic if and only if $\mu=0$.

There exist many unit speed curves $F=re^{i\theta}$ whose curvature satisfies $\kappa=m\theta'$ with $m\in\mathbf{R}$.

Example 1. If $F=re^{i\theta}$ with $r=b^{-1}\cos bs$ and $\theta=bs$, $b>0$, then the curvature of F satisfies $\kappa=2\theta'$. The associated complex extensor is called a *Lagrangian pseudo-sphere.*

Example 2 (Cardioid)**.** Let $F=re^{i\theta}$ be the unit speed reparametrization of $G=(1+\cos t)e^{it}$. Then F satisfies $\kappa(s)=\frac{3}{2}\theta'(s)$.

Example 3 (Circle)**.** Let $F=b^{-1}e^{ibs}$, $b>0$. Then F satisfies $\kappa=\theta'=b$.

Example 4 (Logarithmic spiral)**.** Let $F=(bs/\sqrt{1+b^2})e^{ib^{-1}\ln s}$ with $b>0$. Then F satisfies $\kappa=\theta'=b^{-1}s^{-1}$.

Example 5. Let $F=\sqrt{s^2+b^2}e^{i\tan^{-1}(s/b)}$, $b>0$. Then the curvature of F satisfies $\kappa=0$.

Example 6. Consider $s=iE(\frac{i}{2}\operatorname{arccosh} f;2)$, where $E(\,\cdot\,;k)$ is the elliptic integral of the second kind with elliptic modulus k. Then $s(f)$ is a real-valued decreasing function for $f\ge 1$. If $f(s)$ is its inverse function, then $F=\sqrt{f}e^{i\theta}$ with $\theta=\int_0^s f^{-\frac{3}{2}}ds$ is a unit speed curve satisfying $\kappa=-\theta'$.

3. Hamiltonian-stationary Lagrangian submanifolds. Let ι_0^n denote the inclusion of the unit hypersphere centered at the origin and $F=r(s)e^{i\theta(s)}$ a unit speed curve in $\mathbf{C}^*$ with $\theta'\ne 0$.

Theorem 1. *Let $L: M\to\mathbf{C}^n$ be a Lagrangian H-umbilical submanifold with $n\ge 3$. Then L is Hamiltonian-stationary if and only if, up to dilations, L is congruent to an open portion of a Lagrangian submanifold of the following six types:*

(1) *A Lagrangian cylinder over a circle:*

$$L(s,x_2,\dots,x_n)=\left(\frac{e^{ias}}{a},x_2,\dots,x_n\right),\ a>0.$$

(2) *A complex extensor $F\otimes\iota_0^n$, where F is a unit speed curve whose curvature κ satisfies $\kappa=\theta'(s)$.*

(3) *A complex extensor $F\otimes\iota_0^n$, where F is a unit speed curve with $\kappa=(1-n)\theta'(s)$.*

(4) *A complex extensor $F\otimes\iota_0^n$, where F is a unit speed curve with $\kappa=(3-n)\theta'(s)$.*

(5) *A complex extensor $F\otimes\iota_0^3$, where $F=re^{i\theta}$ is a unit speed curve with $\kappa=br^{-4}$, $b\ne 0$.*

(6) *A complex extensor $F\otimes\iota_0^n, n>3$, where $F=re^{i\theta}$ is a unit speed curve such that the curvature κ satisfies $\kappa\ne m\theta'$ for any $m\in\mathbf{R}$ and*

$$\kappa=\left(\frac{3-n}{2}\right)\theta'+\frac{1}{2(1-n)}\frac{\kappa'}{(\ln r)'}.$$

Proof. Assume $L: M\to\mathbf{C}^n$ is Lagrangian H-umbilical with $n\ge 3$. Then, L is a Lagrangian submanifold without totally geodesic points such that the second fundamental form satisfies (2.7) for some functions λ and μ with respect to some suitable orthonormal local frame field $e_1,\dots,e_n$.

Let $\omega^1,\dots,\omega^n$ denote the dual 1-forms of $e_1,\dots,e_n$ and $(\omega_i^j), i,j=1,\dots,n$, be the connection forms of the Lagrangian submanifold. By applying Codazzi's equation to (2.7), we find

$$(3.1)\qquad e_1\mu=(\lambda-2\mu)\omega_1^j(e_j),\quad j>1,$$

(3.2) $e_j\lambda = (2\mu - \lambda)\omega_j^1(e_1), \ \ j > 1,$

(3.3) $(\lambda - 2\mu)\omega_1^j(e_k) = 0, \ \ 1 < j \neq k \leq n,$

(3.4) $e_j\mu = 3\mu\omega_1^j(e_1),$

(3.5) $\mu\omega_1^j(e_1) = 0, \ \ j > 1.$

It follows from (2.7) that the mean curvature vector H is given by $nH = (\lambda + (n-1)\mu)Je_1$. So, the dual 1-form α_H of JH satisfies

(3.6) $$-n\alpha_H = (\lambda + (n-1)\mu)\omega^1.$$

Now, assume that L is Hamiltonian-stationary. Let δ denote the co-differential operator of M. Since the Hamiltonian-stationary condition of the Lagrangian submanifold in $\mathbf{C}^n$ is characterized by $\delta\alpha_H = 0$ (cf. [15]), so after applying δ to (3.6) and using Cartan's structure equations, we obtain

(3.7) $$e_1\lambda + (n-1)e_1\mu = (\lambda + (n-1)\mu)\sum_{j=2}^{n}\omega_j^1(e_j).$$

Case (A): *M is of constant sectional curvature.* In this case, Theorem 3.1 of [9] implies that either M is an open portion of a Lagrangian pseudo-sphere or M is a flat manifold.

If M is an open portion of a Lagrangian pseudo-sphere, then we have $\lambda = 2\mu$ which is constant on M. Thus, (3.7) reduces to

(3.8) $$\omega_2^1(e_2) + \cdots + \omega_n^1(e_n) = 0 \quad \text{on } U.$$

On the other hand, the Lagrangian pseudo-sphere satisfies $\omega_j^1(e_j) = b\tan bs$ for $j > 1$. Combining this with (3.8) shows that this cannot happen.

If M is flat, it follows from (2.7) and equation of Gauss that $\mu = 0$ identically. Since $\lambda \neq 0$, it follows from (3.1) and $\mu = 0$ that $\omega_j^1(e_j) = 0, j = 2, \ldots, n$. Combining this with (3.3) and (3.7) gives

(3.9) $$e_1\lambda = \omega_j^1(e_k) = 0, \quad 2 \leq j, k \leq n.$$

Also, it follows from (3.2) that

(3.10) $$e_j(\ln\lambda) = \omega_1^j(e_1), \quad j = 2, \ldots, n.$$

Let $\mathcal{D}$ and $\mathcal{D}^\perp$ denote the distributions on M spanned by $\{e_1\}$ and $\{e_2, \ldots, e_n\}$, respectively. Then $\mathcal{D}$ is integrable, since it is 1-dimensional. Also, it follows from (3.9) that $\mathcal{D}^\perp$ is integrable with totally geodesic leaves. Moreover, it follows from (2.7) with $\mu = 0$ that the leaves of $\mathcal{D}^\perp$ are totally geodesic in $\mathbf{C}^n$ as well. Because $\mathcal{D}$ and $\mathcal{D}^\perp$ are both integrable, there exist local coordinates $\{s, x_2, \ldots, x_n\}$ such that $\partial/\partial s$ spans $\mathcal{D}$ and $\{\partial/\partial x_2, \ldots, \partial/\partial x_n\}$ spans $\mathcal{D}^\perp$. Since $\mathcal{D}$ is 1-dimensional, we may choose s in such way that $\partial/\partial s = \lambda^{-1}e_1$.

From $e_1\lambda = 0$, we have $\lambda = \lambda(x_2, \ldots, x_n)$. With respect to $\{s, x_2, \ldots, x_n\}$, (2.7) becomes

(3.11)
$$h\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}\right) = J\,\frac{\partial}{\partial s},\ h\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial x_j}\right) = h\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right) = 0$$

for $j, k = 2, \ldots, n$. Let N^{n-1} be an integral submanifold of $\mathcal{D}^\perp$. Then N^{n-1} is totally geodesic in $\mathbf{C}^n$. Thus, N^{n-1} is an open portion of a Euclidean $(n-1)$-space $\mathbf{E}^{n-1}$. Hence, M is isometric to an open portion of the warped product manifold ${}_{\lambda^{-1}}I \times \mathbf{E}^{n-1}$ with warped product metric:

(3.12) $$g = \lambda^{-2}ds^2 + dx_2^2 + dx_3^2 + \cdots + dx_n^2,$$

where I is an open interval on which λ^{-1} is defined.

Put $\lambda_j = \frac{\partial\lambda}{\partial x_j}$, $\lambda_{jk} = \frac{\partial^2\lambda}{\partial x_j \partial x_k}$ for $j, k = 2, \ldots, n$. From (3.12) we find

(3.13)
$$\nabla_{\frac{\partial}{\partial s}}\frac{\partial}{\partial s} = \sum_{k=2}^{n}\frac{\lambda_k}{\lambda}\frac{\partial}{\partial x_k},\ \ \nabla_{\frac{\partial}{\partial s}}\frac{\partial}{\partial x_j} = -\frac{\lambda_j}{\lambda}\frac{\partial}{\partial s},$$
$$\nabla_{\frac{\partial}{\partial x_j}}\frac{\partial}{\partial x_k} = 0,$$

for $2 \leq j, k \leq n$. By applying (3.13) we find

$$R\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial s} = -\sum_{k=2}^{n}\frac{\lambda_{jk}}{\lambda}\frac{\partial}{\partial x_k}, \quad j = 2, \ldots, n.$$

Since M is flat, this implies that $\lambda_{jk} = 0$ for $j, k = 2, \ldots, n$. Therefore, we have

(3.14) $$\lambda = a + \alpha_2 x_2 + \cdots + \alpha_n x_n,$$

for some $a, \alpha_2, \ldots, \alpha_n \in \mathbf{R}$. From (3.11), (3.13), (3.14) and the formula of Gauss, we obtain

(3.15)
$$L_{ss} = \sum_{k=2}^{n}\frac{\alpha_k}{\lambda}L_{x_k} + iL_s,\ \ L_{sx_j} = -\frac{\alpha_j}{\lambda}L_s,$$
$$L_{x_j x_k} = 0,\ \ j, k > 1.$$

Solving the last equation in (3.15) yields

(3.16) $$L = \sum_{j=2}^{n}P_j(s)x_j + D(s),$$

for some $\mathbf{C}^n$-valued functions $P_2, \ldots, P_n, D$.

By applying (3.14), (3.15) and (3.16), we find

(3.17) $$\alpha_j P_j'(s) = 0,$$

$$(3.18) \quad \alpha_j P_k'(s) + \alpha_k P_j'(s) = 0, \;\; 2 \le j \ne k \le n,$$

$$(3.19) \quad a P_j'(s) + \alpha_j D'(s) = 0, \;\; j, k = 2, \dots, n.$$

If $\alpha_2, \dots, \alpha_n$ are not all zero, say $\alpha_2 \ne 0$. Then, (3.17) gives $P_2' = 0$. Thus, by (3.18) and (3.19), we have $P_3' = \cdots = P_n' = D' = 0$ as well. Hence, $P_2, \dots, P_n$ and D are constant vectors, which is impossible in views of (3.16). Therefore, we must have $\alpha_2 = \cdots = \alpha_n = 0$ and $\lambda = a \ne 0$. So, from (3.19) we know that $P_2, \dots, P_n$ are orthonormal constant vectors in $\mathbf{C}^n$. Consequently, (3.16) becomes

$$(3.20) \quad L = D(x_1) + c_2 x_2 + \cdots + c_n x_n$$

for $c_2, \dots, c_n \in \mathbf{C}^n$. Substituting this into the first equation of (3.15) yields $D(x_1) = c_1 e^{ix_1}$. Hence, L is a Lagrangian cylinder over a circle. Thus, after choosing suitable initial conditions, we get case (1).

Case (B): *M contains no open subset of constant curvature.* By Theorem 4.1 of [9]), M is congruent to a complex extensor $\phi = F \otimes \iota_0^n$ of ι_0^n. Thus, M is a Lagrangian H-umbilical submanifold satisfying (2.7) with $\lambda \ne 2\mu$ and $\mu \ne 0$.

For the complex extensor $F \otimes \iota_0^n$, we have

$$(3.21) \quad \frac{\partial \phi}{\partial s} = F'(s) \otimes \iota_0^n, \;\; e_j \phi = F \otimes e_j, \;\; j > 1.$$

Thus, the metric g of ϕ is given by

$$(3.22) \quad g = ds^2 + f(s) g_1,$$

where $f = \langle F, F \rangle$ and g_1 is the standard metric of the unit n-sphere. As before, we choose $\{e_1, \dots, e_n\}$ with $e_1 = \partial/\partial s$ so that we have (2.7) with

$$(3.23) \quad \lambda = \varphi'(s), \;\; \mu = \frac{\langle F', iF \rangle}{f}, \;\; F'(s) = e^{i\varphi(s)}.$$

Moreover, it follows from (3.22) that

$$(3.24) \quad \omega_2^1(e_2) = \cdots = \omega_n^1(e_n) = -\frac{f'}{2f}.$$

Since $F(s)$ is unit speed, we have

$$(3.25) \quad F'' = i\kappa F', \quad F = \langle F, F' \rangle F' - \langle F', iF \rangle iF',$$

where κ is the curvature of F. It follows from (3.23) and the first equation of (3.25) that

$$(3.26) \quad \lambda = \kappa.$$

From the second equation in (3.25) we find

$$(3.27) \quad 4 \langle F, iF' \rangle^2 = 4f - f'^2 \ge 0.$$

Thus, after replacing s by $-s$ if necessary, we have

$$(3.28) \quad \langle F', iF \rangle = \frac{1}{2}\sqrt{4f - f'^2}.$$

If $4f = f^2$ holds on an open interval I_0, then $\langle F, iF' \rangle = 0$ on I_0. Hence, $F(s)$ is parallel to $F'(s)$ for $s \in I_0$, which implies that $F : I_0 \to \mathbf{C}^*$ is an open part of a line through the origin. So, according to Lemma 2.1, the complex extensor ϕ has totally geodesic points which is a contraction.

From the first equation in (3.25), we find $f'' = 2 - 2\kappa \langle F', iF \rangle$. Combining this with (3.28) yields

$$(3.29) \quad \kappa(s) = \frac{2 - f''(s)}{\sqrt{4f(s) - f'^2(s)}}.$$

Hence, (3.23), (3.26), (3.28) and (3.29) give

$$(3.30) \quad \kappa = \lambda = \frac{2 - f''}{\sqrt{4f - f'^2}}, \;\; \mu = \theta' = \frac{\sqrt{4f - f'^2}}{2f}.$$

Due to $f' = 2\langle F, F' \rangle$ and (3.28), the second equation in (3.25) can be written as

$$(3.31) \quad F'(s) = \frac{f'(s) + i\sqrt{4f(s) - f'^2(s)}}{2f(s)} F(s).$$

Assume f is defined on a open interval $I \ni 0$. After solving (3.31) and using $|F'| = 1$, we know that, up to rotations about the origin, F is given by

$$(3.32) \quad F = \sqrt{f} \exp\left(\frac{i}{2} \int_0^s \frac{\sqrt{4f - f'^2}}{f} ds \right).$$

Since $\mu \ne 0$ and $\lambda \ne 2\mu$, (3.3) and (3.5) give

$$(3.33) \quad \omega_1^j(e_1) = 0, \;\; \omega_j^1(e_j) = \frac{e_1 \mu}{2\mu - \lambda}, \;\; \omega_j^1(e_k) = 0$$

for $2 \le j \ne k \le n$. By substituting the second equation of (3.33) into (3.7) we find

$$(3.34) \quad (2\mu - \lambda)\lambda' = (n-1)(2\lambda + (n-3)\mu)\mu'.$$

So, by combining (3.30) and (3.34) we obtain

$$(3.35) \quad 2(4f - f'^2)(f\psi' + (n-2)f'\psi) + f'\psi^2 = 0,$$

where

$$(3.36) \quad \psi = 2ff'' + (n-3)f'^2 + 4(2-n)f.$$

Since $f' = 2\langle F, F \rangle$ and $f'' = 2 + 2\kappa \langle F, iF' \rangle$, the function ψ can be written as

$$(3.37) \quad \psi = 4(\kappa f - (n-3)\langle F, iF' \rangle) \langle F, iF' \rangle.$$

Case (B.i): *f is a polynomial in s.* A direct computation shows that the only polynomials which satisfy (3.35) are of degree 0 or 2. If f is of degree 2, the leading coefficient must be one. For those polynomials the function ψ in (3.36) is constant.

If f is of degree zero, we may put $f = b^{-2}$ for some $b > 0$. So, from (3.30) we find $\kappa = \theta' = b$, which gives case (2) of the theorem.

If $f = s^2 + bs + c$, then after applying a suitable translation in s, we get $f = s^2 + a$ for some real number a. Thus, by (3.29) we get $\kappa = 0$. Moreover, it is easy to verify that under $f = s^2 + a$, (3.32) holds if and only if either $n = 3$ or $a = 0$. The later case cannot occurs, since the Lagrangian H-umbilical submanifold has no totally geodesic points. Therefore, we obtain case (4) with $n = 3$.

Case (B.ii): *f is not a polynomial in s.* The function ψ given by (3.36) is non-constant. Moreover, (3.24) and (3.33) yields $e_1\mu \neq 0$.

Case (B.ii.a): *$\lambda = m\mu \neq 0$ for some $m \in \mathbf{R}$.* Since $e_1\mu \neq 0$, after substituting $\lambda = m\mu$ into (3.34), we find $(n+m-3)(n+m-1) = 0$, which gives cases (3) and (4).

Case (B.ii.b): *$\lambda \neq c\mu$ for any $c \in \mathbf{R}$.* By applying (3.27), and (3.37), we obtain from (3.35) that

$$2f^2\kappa' = (1-n)f'(2f\kappa + (3-n)\langle F, iF'\rangle). \tag{3.38}$$

From $|F'| = 1$ we have $r^2\theta'^2 + r'^2 = 1$. Without loss of generality, we may assume that $\theta' = r^{-1}\sqrt{1-r'^2}$. Thus, from $F = r(s)e^{i\theta(s)}$ and (3.38), we obtain

$$r\kappa' + (n-1)(2\kappa + (n-3)\theta')r' = 0,$$

which gives case (5) for $n = 3$ and case (6) for $n > 3$.

The converse can be verify by direct computation. □

4. Complex extensors with parallel mean curvature vector.

Theorem 2. *A complex extensor $F \otimes \iota_0^n$ of ι_0^n via a unit speed curve F in $\mathbf{C}^*$ has parallel mean curvature vector if and only if either* (1) *the complex extensor is a minimal Lagrangian submanifold, or* (2) *F is a circle centered at the origin.*

Proof. We already know that the complex extensor $F \otimes \iota_0^n$ is a non-totally geodesic Lagrangian submanifold whose second fundamental form satisfies (2.7) for some functions λ and μ with respect to some suitable orthonormal local frame field $e_1, \ldots, e_n$.

Since the mean curvature vector H is given by

$$H = \frac{1}{n}(\lambda + (n-1)\mu)Je_1, \tag{4.1}$$

the complex extensor ϕ has parallel mean curvature vector if and only if L is minimal or $\lambda + (n-1)\mu$ is a nonzero constant and $\nabla e_1 = 0$.

Now, assume that $F \otimes \iota_0^n$ is non-minimal. Then from $\nabla e_1 = 0$ we have $\omega_1^j(e_k) = 0$ for $j, k = 1, \ldots, n$. Combining this with (3.1) shows that μ is constant.

On the other hand, since $\mu = \frac{1}{2f}\sqrt{4f - f'^2}$, after differentiating μ, we find

$$(ff'' - f'^2 + 2f)f' = 0. \tag{4.2}$$

If $f' = 0$, f is a positive constant. Thus, F is a circle centered at the origin; hence the complex extensor $F \otimes \iota_0^n$ has parallel mean curvature vector.

When $ff'' - f'^2 + 2f = 0$ holds, then after applying a suitable translation in s and replacing s by $-s$ if necessary, we obtain

$$f = s^2,\ f = \frac{4}{b^2}\sinh^2\left(\frac{bs}{2}\right), \text{ or } f = \frac{4}{b^2}\sin^2\left(\frac{bs}{2}\right),$$

according to $c = 0, c = b^2 > 0$, or $c = -b^2 < 0$.

If $f = s^2$, we have $4f = f'^2$. So, the complex extensor is totally geodesic, which is a contradiction.

If $f = \frac{4}{b^2}\sinh^2\left(\frac{bs}{2}\right)$ holds, we get $4f < f'^2$. This is impossible due to (3.27).

If $f = \frac{4}{b^2}\sin^2\left(\frac{bs}{2}\right)$, then we have $\sqrt{4f - f'^2} = \frac{4}{b}\sin^2\left(\frac{bs}{2}\right)$. Thus (3.30) gives $\lambda = 2\mu$. So, $F \otimes \iota_0^n$ is a Lagrangian pseudo-sphere. This is impossible, since $\nabla e_1 \neq 0$ for Lagrangian pseudo-spheres. □

5. Remarks.

Remark 1. If a unit speed curve F satisfies $\kappa = m\theta'(s)$ for some $m \in \mathbf{R}$, then $f = \langle F, F\rangle$ satisfies

$$2ff'' - mf'^2 + 4(m-1)f = 0. \tag{5.1}$$

After solving this differential equation for f' we get

$$4f - f'^2 = \alpha f^m \tag{5.2}$$

for some $\alpha > 0$. Whenever $4f - f'^2 > 0$, we may put $\alpha = 4b^2$, $b > 0$. Thus, if $s(f)$ is an anti-derivative of

$$\frac{1}{2\sqrt{f - b^2 f^m}},$$

the inverse function f of s satisfies (5.1). Thus, by (3.32), we know that $F = \sqrt{f}e^{i\theta}$ with $\theta = \int_0^s bf^{\frac{m}{2}-1}ds$ is a unit speed curve satisfying $\kappa = m\theta'$.

Remark 2. Put $y_1 = f, y_2 = f'$ and $y_3 = f''$. Then equation (3.35) is equivalent to the system:

$$y_1' = y_2,\ \ y_2' = y_3,$$

$$y_3' = \frac{y_2}{4y_1^2(4y_1 - y_2^2)}\{4(4(n-2)n + (n^2 - 4n + 3)y_2^4 - y_3(4n - 8 + y_3))y_1^2 - 4(n-1)(2n - 4 - y_3)y_1y_2^2\}.$$

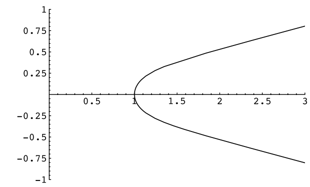

Fig. 1. $\kappa = -5\theta', \theta(0) = 0, r(0) = 1, \varphi(0) = \frac{\pi}{2}$.

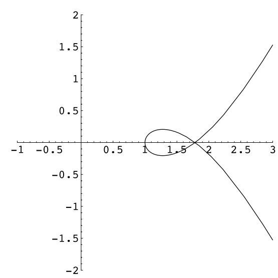

Fig. 2. $\kappa = 6r^{-4}, \theta(0) = 0, r(0) = 1, \varphi(0) = \frac{\pi}{2}$.

It follows from Picard's theorem that, for a given initial conditions: $y_1(s_0) = y_1^0, y_2(s_0) = y_2^0, y_3(s_0) = y_3^0$ at s_0 with $y_1^0 > 0$ and $4y_1^0 > y_2^0$, the initial value problem has a unique solution in some open interval containing s_0. So, (3.35) admits infinitely many positive solutions f with $4f > f'^2$. Each f gives rise to a unit speed curve F whose curvature satisfies

$$r\kappa' + (n-1)(2\kappa + (n-3)\theta')r' = 0.$$

So, there are infinitely many Hamiltonian-stationary Lagrangian submanifolds of type (6) of Theorem 1.

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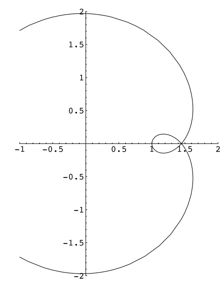

Fig. 3. $\kappa = 8r^{-4}, \theta(0) = 0, r(0) = 1, \varphi(0) = \frac{\pi}{2}$.

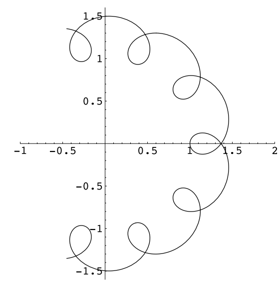

Fig. 4. $\kappa = 9r^{-4}, \theta(0) = 0, r(0) = 1, \varphi(0) = \frac{\pi}{2}$.

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