Iwasawa invariants on non-cyclotomic $\mathbf{Z}_p$-extensions of CM fields

By Hideki Goto

Department of Mathematics, Meiji University, 1–1–1 Tama-ku, Kawasaki, 214–0038 Japan

(Communicated by Shigefumi Mori, M.J.A., Nov. 13, 2006)

Abstract: Let p be an odd prime which splits completely into distinct primes in a CM field K. By considering ray class field of K with respect to prime ideals lying above p, one can define a certain special non-cyclotomic $\mathbf{Z}_p$-extension over K. We will give some examples of such non-cyclotomic $\mathbf{Z}_p$-extensions whose Iwasawa λ- and μ-invariants both vanish, using a variant of a criterion due to Greenberg.

Key words: $\mathbf{Z}_p$-extensions; Iwasawa invariants; Greenberg conjecture.

1. Introduction. Let k be a finite extension of $\mathbf{Q}$ and p be a fixed prime. Let

$$k = k_0 \subset k_1 \subset k_2 \subset \cdots \subset k_n \subset \cdots \subset k_\infty$$

be a cyclotomic $\mathbf{Z}_p$-extension. For arbitrary number field k, Iwasawa has proved that there exist integers μ, λ, and ν such that the power of p dividing the class number of k_n is p^{e_n}, where $e_n = \mu p^n + \lambda n + \nu$ for all sufficiently large n. He also proved that if $k = \mathbf{Q}$, the class numbers of all intermediate fields of $\mathbf{Q}_\infty/\mathbf{Q}$ are prime to p. This was based on a fact that a unique prime of $\mathbf{Q}$ is totally ramified in $\mathbf{Q}_\infty$. From this result, Fukuda and Komatsu [1] considered a non-cyclotomic analogue where the base field was an imaginary quadratic field. They gaved the following criterion:

Let p be an odd prime number which splits into two distinct primes $\mathfrak{p}$ and $\bar{\mathfrak{p}}$ in an imaginary quadratic field K and K_∞ the non-cyclotomic $\mathbf{Z}_p$-extension over K which is constructed through ray class fields with respect to $\mathfrak{p}$. Let K_n be the n-th layer of K_∞, A_n the p-primary part of the ideal class group of K_n, $B_n = \{c \in A_n | c^\sigma = c$ for any $\sigma \in \mathrm{Gal}(K_\infty/K)\}$, and D_n the subgroup of A_n consisting of classes which contain an ideal, all of whose prime factors lie above $\mathfrak{p}$. If $\mathfrak{p}$ totally ramifies in K_∞/K then the Iwasawa invariant $\mu(K_\infty/K)$ and $\lambda(K_\infty/K)$ vanish, if and only if $B_n = D_n$ for some integer $n \geq 0$.

The purpose of this paper is to extend this result in the case where the base field is a general CM field, under some assumptions, we can see that both Iwasawa λ- and μ-invariants vanish. These situation can be also considered as an analog to Greenberg's conjecture which states that both μ and λ vanish for the cyclotomic $\mathbf{Z}_p$-extension of any totally real number field. We hope that studies of this problem provide somewhat new approaches to the conjecture.

We also note that since the number of fundamental units on a totally imaginary quartic field is only one, in this case, we can handle it similarly as in the case of cyclotomic $\mathbf{Z}_p$-extension of a real quadratic field which is well known.

2. CM field. Let K be a finite abelian CM field over $\mathbf{Q}$ of degree $2m$ and let p be a fixed odd prime which splits completely as $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ and $\bar{\mathfrak{p}}_1, \ldots, \bar{\mathfrak{p}}_m$ which is a complex conjugation in K. We denote by $U_{1,\mathfrak{p}_i}$ the pricipal local unit of $\mathfrak{p}_i$ and by $E_{K,1}$ the group which consits of the units in K congruent to 1 modulo all the primes $\mathfrak{p}_i$. Embedding $E_{K,1}$ to $\prod_{i=1}^{m} U_{1,\mathfrak{p}_i}$ diagonally, we denote by $\bar{E}_{K,1}$ the closure of $E_{K,1}$ in that product group.

Let $\tilde{K}$ be the composite of all $\mathbf{Z}_p$-extensions unramified outside $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ and F the maximal abelian pro-p extension of K unramified outside $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$. Then $\tilde{K} \subseteq F$ and it follows from class field theory that $[F : \tilde{K}] < \infty$. Let E_K denote the unit group of K and W_K the group of roots of unity in K. One can show easily that $[E_K : W_K E_{K^+}] \leq 2$, where K^+ is the maximal real subfield of K. Thus Leopoldt's conjecture, which is valid for an abelian number field K^+, states that $\mathrm{rank}_{\mathbf{Z}_p} \bar{E}_{K,1} = \mathrm{rank}_{\mathbf{Z}_p} \bar{E}_{K^+,1} = m - 1$. Recall that $\mathrm{rank}_{\mathbf{Z}_p} \mathrm{Gal}(F/K) = \mathrm{rank}_{\mathbf{Z}_p} (\prod_{i=1}^{m} U_{1,\mathfrak{p}_i}/\bar{E}_{K,1})$, therefore $\mathrm{rank}_{\mathbf{Z}_p} \mathrm{Gal}(F/K) = 1$, which implies that there is a unique $\mathbf{Z}_p$-extension over K unramified outside

2000 Mathematics Subject Classiffcation. Primary 11R23; Secondary 11R29.

$\mathfrak{p}_1, ..., \mathfrak{p}_m$.

Now, we will denote $K'_n = K(\mathfrak{p}_1^{n+1}\mathfrak{p}_2^{n+1}\cdots\mathfrak{p}_m^{n+1})$ the ray class field of K modulo $\mathfrak{p}_1^{n+1}\mathfrak{p}_2^{n+1}\cdots\mathfrak{p}_m^{n+1}$ and define $K'_\infty = \bigcup_{n=0}^{\infty} K'_n$. We had proved that a unique $\mathbf{Z}_p$-extension K_∞ over K exists in K'_∞.

Let K_n be the n-th layer of K_∞/K and A_n the p-primary part of the ideal class group of K_n. Assuming that $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ ramifies totally in K_∞/K, we can use the same criterion of Greenberg [2] to see both Iwasawa invariants $\mu(K_\infty/K)$ and $\lambda(K_\infty/K)$ vanish.

3. Criteria. Let $\Gamma = \mathrm{Gal}(K_\infty/K)$, and we denote by σ a fixed generator of Γ. Let $Cl(\mathfrak{a})$ be the ideal class of A_n for a fractional ideal $\mathfrak{a}$ of K_n. We put,

$$B_n = A_n^\Gamma = \{Cl(\mathfrak{a}) \in A_n | Cl(\mathfrak{a})^\sigma = Cl(\mathfrak{a})\}$$

$$D_n = \{Cl(\mathfrak{a}) \in A_n | \mathfrak{a} = \prod_{i=1}^{m} \mathfrak{P}_i^{a_i},\ \mathfrak{P}_i : \text{prime above } \mathfrak{p}_i,\ a_i \geq 0\}.$$

We can prove the following Lemma and Theorem similarly as the Proposition 1 and Theorem 2 of Greenberg [2].

Lemma 3.1. *Let K be an abelian CM field. Then $|B_n|$ remains bounded as $n \to \infty$.*

Proof. Let L_n denote the maximal unramified abelian p-extension of K_n. By class field theory, $\mathrm{Gal}(L_n/K_n) \cong A_n$. If L'_n denotes the maximal abelian extension of K contained in L_n, one can see easily that L'_n corresponds to $A_n^{\sigma-1}$ and hence that $[L'_n : K_n] = [A_n : A_n^{\sigma-1}] = |B_n|$. On the other hand, since $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ totally ramifies in K_∞/K, $L'_n \cap K_\infty = K_n$. Thus $[L'_n : K_n] = [L'_n K_\infty : K_\infty] \leq [F : K_\infty] < \infty$ □

Theorem 3.2. *Let K be an abelian CM field of degree $2m$ which p splits completely as $\mathfrak{p}_1, \ldots, \mathfrak{p}_m, \bar{\mathfrak{p}}_1, \ldots, \bar{\mathfrak{p}}_m$ on K. Let K_∞ be a unique $\mathbf{Z}_p$-extension over K where the n-th layer exists in the ray class field of K modulo $\mathfrak{p}_1^{n+1}\mathfrak{p}_2^{n+1}. \cdots \mathfrak{p}_m^{n+1}$. Assume that $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ totally ramifies in K_∞/K. Then the following two statements are equivalent:*

1. *$B_n = D_n$ for all sufficiently large integer n.*
2. *$|A_n|$ is bounded as $n \to \infty$.*

Proof. Let $N_{m,n}$ be the norm mapping from K_m to $K_n (m \geq n)$. We first observe that $|A_n|$ is bounded as $n \to \infty$ if and only if for all sufficiently large n and all $m \geq n$, $N_{m,n} : A_m \to A_n$ is an isomorphism. We also observe that $\mathrm{Ker}(N_{m,n}) \neq 1$ if and only if $\mathrm{Ker}(N_{m,n}) \cap B_m \neq 1$. By Lemma 3.1, $|B_n|$ is bounded as $n \to \infty$. It follows that for some n_0, $|B_m| = |B_n|$ for all $m, n \geq n_0$. These remarks imply that $|A_n|$ is bounded as $n \to \infty$ if and only if for all sufficiently large n and all $m \geq n \geq n_0$, $N_{m,n} : B_m \to B_n$ is an isomorphism. Since $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ totally ramifies in K_∞/K, we see that $N_{m,n} : D_m \to D_n$ is surjective for all $m \geq n$. Thus $N_{m,n} : B_m \to B_n$ is surjective and injectivity would follow for $n \geq n_0$. Therefore, 1 implies 2.

Now assume 2. Let B'_n denote the subgroup of A_n consisting of ideal classes containing ideals invariant under the action of $\mathrm{Gal}(K_n/K)$. We will show that $B'_n = B_n$ for sufficiently large n. Let n be such that $N_{m,n} : B_m \to B_n$ is surjective for $m \geq n$. Let $c \in B_n$ and let $\mathfrak{b}$ be an ideal of K_m such that $\mathrm{Cl}(\mathfrak{b}) \in B_m$ and $N_{m,n}(Cl(\mathfrak{b})) = c$. Let $\mathfrak{a} = N_{m,n}(\mathfrak{b})$, so that $\mathfrak{a} \in c$. Now $\mathfrak{b}^{\sigma-1} = (\beta)$, where $\beta \in K_m$ and $\mathfrak{a}^{\sigma-1} = (\alpha)$, where $\alpha = N_{m,n}(\beta) \in K_n$. Let $\epsilon = N_{n,0}(\alpha) = N_{m,0}(\beta)$. Then $\epsilon \in E_K$. Now for any prime of K lying above p, $K_{\mathfrak{p}_i}$, the completion of K at $\mathfrak{p}_i$, is isomorphic to $\mathbf{Q}_p$. Thus, by local class field theory, $\epsilon \in N_{m,0}(K_m^\times)$ implies that ϵ is an $\mathfrak{p}_i$-adic p^mth power for all $\mathfrak{p}_i$ which totally ramifies. On the other hand, since $\epsilon^2 \in K^+$, ϵ^2 is an $\bar{\mathfrak{p}}_i$-adic p^mth power for all such $\bar{\mathfrak{p}}_i$. Thus Leopoldt's conjecture implies that $\epsilon \in E_K^{p^n}$ for sufficiently large m, since p is odd. We may assume that $\epsilon = \eta^{p^n}$, where $\eta \in E_K$. Therefore, $\mathfrak{a}^{\sigma-1} = (\alpha) = (\alpha\eta^{-1})$, where $N_{n,0}(\alpha\eta^{-1}) = 1$ and so $\alpha\eta^{-1} = \gamma^{\sigma-1}$for some $\gamma \in K_n$. Hence $\mathfrak{a}(\gamma^{-1}) \in c$. Thus $c \in B'_n$ and $B_n = B'_n$ for sufficiently large n. Recall that $B'_n = D_n \cdot i_{0,n}(A_0)$, where $i_{0,n} : A_0 \to A_n$ denotes a homomorphism sending $\mathrm{Cl}(\mathfrak{a})$ to $\mathrm{Cl}(\mathfrak{a}\mathfrak{O}_{K_n})$ for every ideal $\mathfrak{a}$ of K and $\mathfrak{O}_{K_n}$ the ring of integers of K_n. Statement 2 implies that $i_{0,n}(A_0) = 1$ for n sufficiently large, thus statement 1 will clearly follow. □

4. The order of $\mathbf{B_n}$. In this section we will introduce Inatomi's [3] results which are useful in computing the order of B_n.

Let E_K be the unit group of K as above. $E_K = W_K E'$, where E' is a free abelian group of rank m−1. Let $\zeta = \{\zeta_1, \zeta_2, \cdots, \zeta_{m-1}\}$ be the base of E'. Since $K_{\mathfrak{p}_i} \cong \mathbf{Q}_p (1 \leq i \leq m)$, for any ζ_j there exists a positive integer $m_{\zeta_j,i}$ for $\mathfrak{p}_i$ such that

$$\zeta_j^{p-1} \equiv 1 \,(\mathrm{mod}\ \mathfrak{p}_i^{m_{\zeta_j,i}}) \ \text{ and } \ \zeta_j^{p-1} \not\equiv 1 \,(\mathrm{mod}\ \mathfrak{p}_i^{m_{\zeta_j,i}+1}).$$

We will define $m_{\zeta_j} = \min\{m_{\zeta_j,i} | 1 \leq i \leq m\}$ and $m(\zeta) = m_{\zeta_1} + m_{\zeta_2} + \cdots + m_{\zeta_{m-1}}$. Let $M = \max\{m(\zeta) | \zeta \text{ is a base of } E'\}$ and $|A_K| = p^c$. Then

Table 1. Examples

p	l	m	p	l	m	p	l	m
7	19	−19	11	43	−17	11	211	−13
7	19	−26	11	43	−21	11	211	−21
7	73	−10	11	43	−39	11	211	−35
7	73	−17	11	61	−6	11	229	−6
7	73	−38	11	61	−17	11	229	−7
7	157	−31	11	61	−19	11	229	−10
7	157	−38	11	61	−30	11	229	−17
7	181	−6	11	61	−39	11	229	−21
7	223	−6	11	193	−6	11	229	−29
7	223	−13	11	193	−17	11	229	−39
11	19	−7	11	193	−19	13	103	−10
11	19	−19	11	193	−30	13	103	−14
11	19	−30	11	193	−39	13	103	−17
11	37	−10	11	199	−6	13	103	−23
11	37	−13	11	199	−13	13	103	−29
11	37	−19	11	199	−17	13	103	−30
11	37	−21	11	199	−19	13	163	−10
11	37	−30	11	199	−30	13	163	−23
11	37	−35	11	199	−35	13	163	−30
11	43	−6	11	199	−39	13	163	−35
11	43	−10	11	211	−10	13	193	−10

we may prove the following result by the same way as in the proposition of Inatomi [3].

Proposition 4.1. *Let K_∞ be the non-cyclotomic $\mathbf{Z}_p$-extension over K which satisfy the same condition on Section 3. Then $|B_n| = p^{c+M-(m-1)}$, for $n \geq M$.*

We also note that taking a base of $N_{n,0}(K_n) \cap E'$, we can show that $m(\zeta) = M$ for $n \geq M$.

5. Examples.

Example 5.1. Let K be a composite of an imaginary quadratic field and a cubic cyclic field. Let K_∞ be the non-cyclotomic $\mathbf{Z}_p$-extension over K which satisfy the same condition on Section 3. Assume $|A_K| = 1$, which implies that $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3$ are totaly ramified in K_∞/K. The fact $|B_n| = 1$ implies $|A_n| = 1$. Hence if $M = 2$, namely, $m_{\zeta_1} = m_{\zeta_2} = 1$ for any base $\{\zeta_1, \zeta_2\}$ of E', then we have $|A_n| = 1$ for all $n \geq 0$, which means $\lambda = \mu = \nu = 0$.

Let $\mathbf{Q}(\sqrt{m}), m < 0$ be the imaginary quadratic field and $\mathbf{Q}(\alpha)$ the cubic cyclic field. We may obtain α using the trace due to $\mathrm{Gal}(\mathbf{Q}(\mu_l)/\mathbf{Q}(\alpha))$ for some l with μ_l a primitive lth root of unity. Then the table below are examples considered on $1 \leq l \leq 1000$, $-100 \leq m \leq -1$ which satisfy our condition.

References

[1] T. Fukuda and K. Komatsu, Noncyclotomic $\mathbf{Z}_p$-extensions of imaginary quadratic fields, Experiment. Math. **11** (2002), no. 4, 469–475 (2003).

[2] R. Greenberg, On the Iwasawa invariants of totally real number fields, Amer. J. Math. **98** (1976), no. 1, 263–284.

[3] A. Inatomi, On $\mathbf{Z}_p$-extensions of real abelian fields, Kodai Math. J. **12** (1989), no. 3, 420–422.