Rational solutions of the A_4 Painlevé equation

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Abstract: In this note, we will completely classify all of rational solutions of the A_4 Painlevé equation, which is a generalization of the fourth Painlevé equation. The rational solutions are classified to three types by the Bäcklund transformation group.

Key words: The A_4 Painlevé equation; affine Weyl groups; Bäcklund transformations; rational solutions.

1. Introduction. In this note, we will completely classify all of rational solutions of the A_4 Painlevé equation. The A_4 Painlevé equation is a member of the family of the A_n Painlevé equations found by Noumi-Yamada [6]. The $A_2^{(1)}$ and $A_3^{(1)}$ Painlevé equations correspond to the fourth and the fifth Painlevé equations, respectively. The A_4 Painlevé equation is the first equation of the A_n Painlevé equations, which is not the original Painlevé equations.

The solutions of the Painlevé equations are generally transcendental. But it is known that the Painlevé equations have special solutions which can be expressed by algebraic functions or solutions of linear equations. Rational solutions of all types of the Painlevé equations are completely classified now. Any solution of the first Painlevé equation is transcendental. For the second Painlevé equation, a necessary condition for the existence of rational solutions is found by [8] and Vorob'ev [7] showed that Yablonskii's condition is sufficient in 1965. For the other types of the Painlevé equations, rational solutions are classified in 1977-2000 [2, 1, 5, 3, 4].

The Painlevé equations have the Bäcklund transformations, which transform a solution to another solution of the same equation with different parameters. It is shown by Okamoto that the Bäcklund transformation groups are isomorophic to the affine Weyl groups. For $P_{II}, P_{III}, P_{IV}, P_V, P_{VI}$, the Bäcklund transformation groups correspond to $A_1^{(1)}, A_1^{(1)} \bigoplus A_1^{(1)}, A_2^{(1)}, A_3^{(3)}, D_4^{(1)}$, respectively.

Nowadays, the Painlevé equations are extended in many different ways. Noumi-Yamada discovered the equations of type $A_l^{(1)}$, whose Bäcklund transformation groups are isomorphic to $\tilde{W}(A_l^{(1)})$. In this note, we deal with the equation of type $A_4^{(1)}$ and call it the A_4 Painlevé equation.

$$(A_4)\qquad \begin{aligned} &f_i' = f_i(f_{i+1} - f_{i+2} + f_{i+3} - f_{i+4}) + \alpha_i, \\ &\qquad\qquad\qquad (i = 0, 1, 2, 3, 4) \\ &f_0 + f_1 + f_2 + f_3 + f_4 = t, \end{aligned}$$

where $'$ is the differentiation with respect to t. We consider the suffix of f_i and α_i as elements of $\mathbf{Z}/5\mathbf{Z}$. From (A_4), we have $\sum_{i=0}^4 \alpha_i = 1$. The equation (A_4) is essentially a nonlinear equation with the fourth order.

The Bäcklund transformation group of (A_4) is generated by s_0, s_1, s_2, s_3, s_4 and π:

$$\begin{aligned} &s_i(\alpha_i) = -\alpha_i,\ s_i(\alpha_j) = \alpha_j + \alpha_i\ (j = i \pm 1), \\ &s_i(\alpha_j) = \alpha_j\ (j \neq i, i \pm 1), \\ &s_i(f_i) = f_i,\ s_i(f_j) = f_j \pm \frac{\alpha_i}{f_i}\ (j = i \pm 1), \\ &s_i(f_j) = f_j\ (j \neq i, i \pm 1), \end{aligned}$$

for $i = 0, 1, 2, 3, 4$ and

$$\pi(\alpha_j) = \alpha_{j+1},\ \pi(f_j) = f_{j+1}\ (0 \leq j \leq 4).$$

The Bäcklund transformation group $\langle s_0, s_1, s_2, s_3, s_4, \pi\rangle$ is isomorphic to the extended affine Weyl group $\tilde{W}(A_4^{(1)})$.

We will completely classify rational solutions of the A_4 Painlevé equation. The result is that rational solutions are decomposed to three classes, each of which is an orbit by the action of $\tilde{W}(A_4^{(1)})$. The details will be republished elsewhere.

Our main theorem is

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Theorem 1.1. *The A_4 Painlevé equation has a unique rational solution if and only if the parameters $(\alpha_j)_{0\le j\le 4}$ satisfy one of the following three conditions.*

(1) $\alpha_0, \alpha_1, \ldots, \alpha_4 \in \mathbf{Z}$.

(2) *For some* $i = 0, 1, \ldots 4$,

$$(\alpha_i, \alpha_{i+1}, \alpha_{i+2}, \alpha_{i+3}, \alpha_{i+4}) = \begin{cases} \pm\frac{1}{3}(1,1,1,0,0) & \bmod \mathbf{Z} \\ \pm\frac{1}{3}(1,-1,-1,1,0) & \bmod \mathbf{Z}. \end{cases}$$

(3) *For some* $i = 0, 1, \ldots, 4$,

$$(\alpha_i, \alpha_{i+1}, \alpha_{i+2}, \alpha_{i+3}, \alpha_{i+4}) = \begin{cases} \frac{j}{5}(1,1,1,1,1) & \bmod \mathbf{Z} \\ \frac{j}{5}(1,2,1,3,3) & \bmod \mathbf{Z}, \end{cases}$$

with some $j = 1, 2, 3, 4$.

(4) *Furthermore, by a suitable Bäcklund transformation, the rational solution of f_j in the class* (1), (2), (3) *above is respectively transformed to the following.*

(1) $(f_0, f_1, f_2, f_3, f_4) = (t, 0, 0, 0, 0)$, *with* $(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = (1, 0, 0, 0, 0)$,

(2) $(f_0, f_1, f_2, f_3, f_4) = \left(\frac{t}{3}, \frac{t}{3}, \frac{t}{3}, 0, 0\right)$, *with* $(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0\right)$,

(3) $(f_0, f_1, f_2, f_3, f_4) = \left(\frac{t}{5}, \frac{t}{5}, \frac{t}{5}, \frac{t}{5}, \frac{t}{5}\right)$, *with* $(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = \left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}\right)$.

We will sketch the proof of Theorem 1.1. Firstly, we will get a necessary condition of the parameters by comparing the residues of f_j. Secondly, we will transfer them to the fundamental domain of $\tilde{W}(A_4^{(1)})$ by Bäcklund transformations. Lastly, we obtain a sufficient condition by comparing the residues of an auxiliary function G (see (4.1)).

2. Necessary condition. In this section, we will give a necessary condition for the A_4 Painlevé equation to have rational solutions. We will compare the residues of f_j. If $(f_j)_{0\le j\le 4}$ is a rational solution of the A_4 Painlevé equation, f_j is expanded at $t = c \in \mathbf{C}, \infty$ in the following way.

Proposition 2.1. *If $(f_j)_{0\le j\le 4}$ is a rational solution of the A_4 Painlevé equation and f_j have poles at $c \in \mathbf{C}$, then c is a simple pole and the residues of f_j are given by one of the following*:

(1)
$$(\mathrm{Res}_c f_i, \mathrm{Res}_c f_{i+1}, \mathrm{Res}_c f_{i+2}, \mathrm{Res}_c f_{i+3}, \mathrm{Res}_c f_{i+4}) = (1, -1, 0, 0, 0) \quad (\textit{for some } i = 0, \ldots, 4).$$

(2)
$$(\mathrm{Res}_c f_i, \mathrm{Res}_c f_{i+1}, \mathrm{Res}_c f_{i+2}, \mathrm{Res}_c f_{i+3}, \mathrm{Res}_c f_{i+4}) = (-1, 0, 1, 0, 0) \quad (\textit{for some } i = 0, \ldots, 4).$$

(3)
$$(\mathrm{Res}_c f_i, \mathrm{Res}_c f_{i+1}, \mathrm{Res}_c f_{i+2}, \mathrm{Res}_c f_{i+3}, \mathrm{Res}_c f_{i+4}) = (0, 3, 1, -1, -3) \quad (\textit{for some } i = 0, \ldots, 4).$$

Proposition 2.2. *Assume that $(f_j)_{0\le j\le 4}$ is a rational solution of the A_4 Painlevé equation. Then the Laurent expansion of f_j at $t = \infty$ must be one of the following. Furthermore its coefficients are determined inductively.*

Type A i) *For some* $i = 0, \ldots, 4$,

$$f_i = t + (-\alpha_{i+1} + \alpha_{i+2} - \alpha_{i+3} + \alpha_{i+4})t^{-1} + O(t^{-2}),$$
$$f_{i+k} = (-1)^{k+1}\alpha_{i+k}t^{-1} + O(t^{-2}) \quad (1 \le k \le 4).$$

Type A ii) *For some* $i = 0, \ldots, 4$,

$$f_i = t + (-1 + \alpha_i - 2\alpha_{i+2} + 2\alpha_{i+4})t^{-1} + O(t^{-2}),$$
$$f_{i+1} = t + (1 - \alpha_{i+1} - 2\alpha_{i+2} + 2\alpha_{i+4})t^{-1} + O(t^{-2}),$$
$$f_{i+2} = \alpha_{i+2}t^{-1} + O(t^{-2}),$$
$$f_{i+3} = -t + (-3\alpha_{i+4} - \alpha_i + \alpha_{i+1} + 3\alpha_{i+2})t^{-1} + O(t^{-2}),$$
$$f_{i+4} = -\alpha_{i+4}t^{-1} + O(t^{-2}).$$

Type B *For some* $i = 0, \ldots, 4$,

$$f_i = t/3 + (\alpha_{i+1} - \alpha_{i+2} - 3\alpha_{i+3} - \alpha_{i+4})t^{-1} + O(t^{-2}),$$
$$f_{i+1} = t/3 + (\alpha_{i+2} - \alpha_i - \alpha_{i+3} + \alpha_{i+4})t^{-1} + O(t^{-2}),$$
$$f_{i+2} = t/3 + (\alpha_i - \alpha_{i+1} + \alpha_{i+3} + 3\alpha_{i+4})t^{-1} + O(t^{-2}),$$
$$f_{i+3} = 3\alpha_{i+3}t^{-1} + O(t^{-2}),$$
$$f_{i+4} = -3\alpha_{i+4}t^{-1} + O(t^{-2}).$$

Type C

$$f_k = t/5 + (3\alpha_{k+1} + \alpha_{k+2} - \alpha_{k+3} - 3\alpha_{k+4})t^{-1} + O(t^{-2}) \quad (0 \le k \le 4).$$

Proposition 2.3. *Assume that $(f_i)_{0\le i\le 4}$ is a rational solution of the A_4 Painlevé equation. Then f_i are odd functions.*

Proof. The map

$$(f_i(t))_{0\le i\le 4} \longrightarrow (-f_j(-t))_{0\le i\le 4}$$

preserves (A_4). This map keeps the Types in Proposition 2.2 and the parameters $(\alpha_i)_{0\le i\le 4}$. From the uniqueness of the coefficients of the Laurent expansions of f_i at $t = \infty$, $f_i(t) = -f_i(-t)$. Therefore, f_i are odd functions. □

By comparing the residues of f_j, we obtain a necessary condition of the parameters where

$(f_j)_{0\le j\le 4}$ is a rational solution of (A_4).

Theorem 2.4. *Assume that the A_4 Painlevé has a rational solution. Then it is necessary that the parameters $(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ satisfy one of the following conditions. (Here we identify the parameters that are transformed to each other by the Bäcklund transformation π.)*

(1) (n_0,n_1,n_2,n_3,n_4) *with* $n_0,\dots,n_4\in\mathbf{Z}$,

(2) $\left(\frac{n_1}{3}-\frac{n_3}{3},\frac{n_1}{3},\frac{n_1}{3}+\frac{n_4}{3},\frac{n_3}{3},-\frac{n_4}{3}\right)\mod\mathbf{Z}$ *with* $n_1,n_3,n_4\in\{0,1,2\}$,

(3) $\left(\frac{n_1}{5}+\frac{2n_2}{5}+\frac{3n_3}{5},\frac{n_1}{5}+\frac{2n_2}{5}+\frac{n_3}{5},\frac{n_1}{5},\frac{n_1}{5}+\frac{n_2}{5},\frac{n_1}{5}+\frac{n_3}{5}\right)\mod\mathbf{Z}$ *with* $n_1,n_2,n_3\in\{0,1,2,3,4\}$.

Type A, B *and* C *correspond to* (1), (2) *and* (3), *respectively.*

3. Fundamental domain. We will transfer the parameters given in Theorem 2.4 to the fundamental domain of the affine Weyl group $\tilde{W}(A_4^{(1)})$ by Bäcklund transformations of the A_4 Painlevé equation.

Theorem 3.1. I) *The parameters in Theorem* 2.4 (1) *are transformed to* $(1,0,0,0,0)$.

II) *The parameters in Theorem* 2.4 (2) *are transformed to one of*

$$\left(\frac{1}{3},\frac{1}{3},\frac{1}{3},0,0\right),\ \left(\frac{2}{3},0,0,\frac{1}{3},0\right),\ \left(\frac{1}{3},0,0,\frac{2}{3},0\right),$$
$$\left(0,\frac{1}{3},0,\frac{1}{3},\frac{1}{3}\right),\ \left(0,\frac{1}{3},\frac{1}{3},\frac{1}{3},0\right),\ \left(0,0,\frac{1}{3},0,\frac{2}{3}\right),$$
$$\left(0,0,\frac{2}{3},0,\frac{1}{3}\right),\ (1,0,0,0,0).$$

III) *The parameters in Theorem* 2.4 (3) *are transformed to one of*

$$\left(\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5}\right),\ (1,0,0,0,0),\ \left(\frac{3}{5},0,\frac{1}{5},\frac{1}{5},0\right),$$
$$\left(\frac{1}{5},0,\frac{2}{5},\frac{2}{5},0\right),\ \left(\frac{1}{5},\frac{2}{5},0,0,\frac{2}{5}\right),\ \left(\frac{3}{5},\frac{1}{5},0,0,\frac{1}{5}\right).$$

The case (1) and the following cases are important because the other cases do not give rational solutions.

Proposition 3.2. *If the parameters in Theorem* 2.4 (2) *transformed to* $\left(\frac{1}{3},\frac{1}{3},\frac{1}{3},0,0\right)$,

$$(\alpha_i,\alpha_{i+1},\alpha_{i+2},\alpha_{i+3},\alpha_{i+4}) = \begin{cases} \pm\dfrac{1}{3}(1,1,1,0,0) & \mod\mathbf{Z} \\ \pm\dfrac{1}{3}(1,-1,-1,1,0) & \mod\mathbf{Z}, \end{cases}$$

holds for some $i=0\dots,4$.

If the parameters in Theorem 2.4 (3) *transformed to* $\left(\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5}\right)$,

$$(\alpha_i,\alpha_{i+1},\alpha_{i+2},\alpha_{i+3},\alpha_{i+4}) = \begin{cases} \dfrac{j}{5}(1,1,1,1,1) & \mod\mathbf{Z} \\ \dfrac{j}{5}(1,2,1,3,3) & \mod\mathbf{Z}, \end{cases}$$

holds for some $i=0,\dots,4$.

4. Sufficient condition. In this section, we will give a sufficient condition for the A_4 Painlevé equation to have a rational solution. We will check the fifteen cases given in Theorem 3.1. From Proposition 4.1 below, we can prove the main theorem.

Proposition 4.1. *Among the fifteen cases given in Theorem* 3.1, *rational solutions exist only for the following three cases*:

Case I, *that is,*
$(\alpha_0,\alpha_1,\alpha_2,\alpha_3.\alpha_4)=(1,0,0,0,0)$,
$(f_0,f_1,f_2,f_3,f_4)=(t,0,0,0,0)$,
$(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4)=\left(\frac{1}{3},\frac{1}{3},\frac{1}{3},0,0\right)$ *in Case* II,
$(f_0,f_1,f_2,f_3,f_4)=\left(\frac{1}{3}t,\frac{1}{3}t,\frac{1}{3}t,0,0\right)$,
$(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4)=\left(\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5},\frac{1}{5}\right)$ *in Case* III,
$(f_0,f_1,f_2,f_3,f_4)=\left(\frac{1}{5}t,\frac{1}{5}t,\frac{1}{5}t,\frac{1}{5}t,\frac{1}{5}t\right)$.

Proof. In the above three cases, we obtain rational solutions easily. We will consider the other twelve cases.

We take the following auxiliary function:

$$(4.1)\quad G=f_0f_1f_2+f_1f_2f_3+f_2f_3f_4+f_3f_4f_0+f_4f_0f_1.$$

If G has a pole at $t=c\in\mathbf{C}$, $\mathrm{Res}_c\,G$ is $\alpha_{i+2}+\alpha_{i+4}$, α_i or $\alpha_{i+1}+\alpha_{i+4}$, for some $i=0,\dots,4$. Therefore, $\mathrm{Res}_c\,G$ is non-negative. Since it is easily checked that $\mathrm{Res}_\infty\,G$ is positive except for the case $(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4)=\left(0,\frac{1}{3},\frac{1}{3},\frac{1}{3},0\right)$, (A_4) does not have rational solutions in the eleven cases other than $\left(0,\frac{1}{3},\frac{1}{3},\frac{1}{3},0\right)$.

We will consider the case of $\left(0,\frac{1}{3},\frac{1}{3},\frac{1}{3},0\right)$. In this case, $\mathrm{Res}_\infty\,G=0$ and we have:

$$\begin{aligned} f_0 &= \frac{1}{3}t-t^{-1}-6t^{-3}-90t^{-5}+\cdots, \\ f_1 &= \frac{1}{3}t, \\ f_2 &= \frac{1}{3}t, \\ f_3 &= \phantom{\frac{1}{3}t-{}} t^{-1}+6t^{-3}+90t^{-5}+\cdots, \\ f_4 &\equiv 0. \end{aligned}$$

If f_0 and f_3 have a pole at $c \in \mathbf{C}$, $\mathrm{Res}_c\ f_3 = -1$ and $\mathrm{Res}_c\ f_0 = 1$ from Proposition 2.1 (2). If $c \in \mathbf{C} \setminus \{0\}$ is a pole of f_0 and f_3, $-c$ is also a pole since f_0 and f_3 are odd by Proposition 2.3. Because $\mathrm{Res}_\infty\ f_0$ and $\mathrm{Res}_\infty\ f_3$ are odd integers, $t = 0$ is a pole of f_0 and f_3. Therefore, f_0 and f_3 are expressed as

$$f_3 = \frac{-1}{t} + \sum_{j=1}^{k} \left(\frac{-1}{t - c_j} + \frac{-1}{t + c_j} \right),$$

$$f_0 = \frac{1}{3}t + \frac{1}{t} + \sum_{j=1}^{k} \left(\frac{1}{t - c_j} + \frac{1}{t + c_j} \right).$$

Since the coefficient of t^{-1} in f_3 is negative and the coefficient of t^{-1} in f_0 is positive, it contradicts the Laurent expansions of f_0 and f_3 at $t = \infty$. □

Remarks. Murata [5] used the analysis of the Riccati equation to obtain the sufficient condition. But we do not use the analysis of the Riccati equation. In the case $f_3 = f_4 = 0$ and $\alpha_3 = \alpha_4 = 0$, (A_4) is equivalent to the fourth Painlevé equation. Therefore, we also showed that rational solutions of the fourth Painlevé equation can be classified only by the method of residue calculus.

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