A new solution of the fourth Painlevé equation with a solvable monodromy

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Abstract: We will study a new special solution of the fourth Painlevé equation, for which we can calculate the linear monodromy exactly. We will show the relation between Umemura's classical solutions and our solutions.

Key words: The Painlevé equation; monodromy data.

1. Introduction. The Painlevé equation can be represented by an isomonodromic deformation of a linear equation. We call the monodromy data of the linear equation **a linear monodromy** of the Painlevé function. The linear monodromy cannot be calculated except for special cases. One exceptional case is Umemura's classical solutions. Umemura showed that there exist two kinds of special solutions for the Painlevé equations, rational solutions and the Riccati solutions [14], which are called classical solutions of the Painlevé equations. For most of all Umemura's classical solutions, the linear monodromy can be calculated, but there exist some Painlevé functions which are not included in Umemura's classical solutions, such that the linear monodromy can be calculated. In this paper we call such Painlevé functions **monodromy solvable**.

It was R. Fuchs who found a monodromy solvable solution at first, which is not included in Umemura's classical solutions [2]. He calculated the linear monodromy of so-called Picard's solutions, which satisfies the sixth Painlevé equation with a special parameter. This result was discovered again recently [8, 9]. Another monodromy solvable solution is a symmetric solution of the first and second Painlevé equation which are shown by A. V. Kitaev [7].

In this paper we construct a monodromy solvable solution for the fourth Painlevé equation in accordance with Kitaev's method. Umemura's special solutions exist only for special values of parameters but our new special solution exists for any value of parameters and the associated linear equation can be reduced to the Whittaker equation for the special initial condition. This solution includes the rational solution $y = -2t/3$ for parameters $(\alpha, \beta) = (0, -2/9)$. Our solution also includes one point of the Riccati solution. In section three we describe the relations between our new solution and Umemura's classicial solutions.

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2. Linear problem.

2.1. Isomonodromic deformation equations. The fourth Painlevé equation

$$(2.1)\qquad P_{IV}:\quad \frac{d^2y}{dt^2} = \frac{1}{2y}\left(\frac{dy}{dt}\right)^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2-\alpha)y + \frac{\beta}{y}$$

is given by isomonodromic deformation equations [6]:

$$(2.2)\qquad \frac{\partial Y(x,t)}{\partial x} = A(x,t)Y(x,t),$$

$$A(x,t) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} x + \begin{pmatrix} t & u \\ \frac{2}{u}(z-\theta_0-\theta_\infty) & -t \end{pmatrix} + \frac{1}{x}\begin{pmatrix} -z+\theta_0 & -\frac{uy}{2} \\ \frac{2z}{uy}(z-2\theta_0) & z-\theta_0 \end{pmatrix},$$

$$(2.3)\qquad \frac{\partial Y(x,t)}{\partial t} = B(x,t)Y(x,t),$$

$$B(x,t) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} x + \begin{pmatrix} 0 & u \\ \frac{2}{u}(z-\theta_0-\theta_\infty) & 0 \end{pmatrix},$$

where y, z and u are functions of t, and θ_0 and θ_∞ are constants

$$(2.4)\qquad \alpha = 2\theta_\infty - 1, \qquad \beta = -8\theta_0^2.$$

Setting $w = z/y$, integrability condition gives

$$\frac{dy}{dt} = -4yw + y^2 + 2ty + 4\theta_0, \tag{2.5}$$

$$\frac{dw}{dt} = 2w^2 - 2yw - 2tw + (\theta_0 + \theta_\infty), \tag{2.6}$$

$$\frac{d\log u}{dt} = -y - 2t. \tag{2.7}$$

The system (2.5) and (2.6) is the Hamiltonian system with the polynomial Hamiltonian H_4:

$$H_4 = -2yw^2 + y^2w + 2tyw + 4\theta_0 w - (\theta_0 + \theta_\infty)y. \tag{2.8}$$

The function u can be obtained from (2.7) by a quadrature.

The solutions of (2.5) and (2.6) with initial data $y(0) = 0$ and $w(0) = 0$ are expanded as follows:

$$y = 4\theta_0 t \sum_{k=0}^{\infty} a_k t^{2k}, \tag{2.9}$$

$$a_0 = 1, \quad a_1 = \frac{-2}{3}(2\theta_\infty - 1),$$

$$a_2 = \frac{1}{30}\{4(2\theta_\infty - 1)^2 + 3(4\theta_0)^2 + 8(4\theta_0) + 4\}, \ldots,$$

$$w = (\theta_0 + \theta_\infty) t \sum_{k=0}^{\infty} b_k t^{2k}, \tag{2.10}$$

$$b_0 = 1, \quad b_1 = \frac{2}{3}(\theta_\infty - 3\theta_0 - 1),$$

$$b_2 = \frac{4}{15}\{(\theta_\infty - 3\theta_0 - 1)^2 + 4\theta_0(2\theta_\infty - 1)\}, \ldots .$$

These solutions are invariant for the transformation acting on (2.5) and (2.6): $y \to -y$, $w \to -w$, $t \to -t$. We call (2.9) and (2.10) hereafter **the symmetric solution of P_{IV}**. By the Painlevé property the symmetric solution are meromorphic over the complex plane. We will study the behavior of the symmetric solution at infinity and the connection problem in the succeeding paper.

2.2. Transformation of the linear equation. By putting $t = 0$, $y = 0$, and $w = 0$ in equation (2.2), we have

$$\frac{d}{dx}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x + \frac{\theta_0}{x} & u \\ \frac{-2(\theta_0+\theta_\infty)}{u} & -x - \frac{\theta_0}{x} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \tag{2.11}$$

By the transformation $x^2 = \xi$ and $y_i = \xi^{\frac{-1}{4}} v_i$, $(i = 1, 2)$, we have the Whittaker equations:

$$\frac{d^2 v_1}{d\xi^2} + \left[\frac{-1}{4} + \frac{k}{\xi} + \frac{\frac{1}{4} - m^2}{\xi^2}\right] v_1 = 0, \tag{2.12}$$

$$\frac{d^2 v_2}{d\xi^2} + \left[\frac{-1}{4} + \frac{k+\frac{1}{2}}{\xi} + \frac{\frac{1}{4} - (m+\frac{1}{2})^2}{\xi^2}\right] v_2 = 0, \tag{2.13}$$

$$k = \frac{2\theta_\infty - 1}{4}, \qquad m = \frac{2\theta_0 - 1}{4}. \tag{2.14}$$

The above discussion proves the following:

Theorem 1. *The symmetric solution* (2.9) *and* (2.10) *of the fourth Painlevé equation is monodromy solvable. For* (2.9) *and* (2.10), (2.2) *is reduced to the Whittaker equation when* $t = 0$. *The solution of* (2.11) *is given by*

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} L_{k,m}(x) & L_{k,-m}(x) \\ \frac{-2k-2m-1}{u(2m+1)} L_{k+\frac{1}{2},m+\frac{1}{2}}(x) & \frac{-4m}{u} L_{k+\frac{1}{2},-m-\frac{1}{2}}(x) \end{pmatrix}, \tag{2.15}$$

where

$$L_{k,m}(x) = x^{2m+\frac{1}{2}} e^{-\frac{x^2}{2}} {}_1F_1\left(m - k + \frac{1}{2}, 2m+1; x^2\right) \tag{2.16}$$

$$= x^{2m+\frac{1}{2}} e^{-\frac{x^2}{2}} \sum_{n=0}^{\infty} \frac{\Gamma(2m+1)\Gamma\left(m-k+\frac{1}{2}+n\right) x^{2n}}{\Gamma(2m+1+n)\Gamma\left(m-k+\frac{1}{2}\right) n!}. \tag{2.17}$$

2.3. The linear monodromy. The equation (2.2) has a regular singular point $x = 0$ and an irregular singular point $x = \infty$ with the Poincaré rank 2. We will define the linear monodromy $\{M_0,\ \Gamma,\ G_1,\ G_2,\ G_3,\ G_4,\ e^{2\pi i T_0}\}$ of (2.2) [3, 6].

1) At the regular sigularity $x = 0$, the local behavior of $Y(x)$ is given by

$$Y^{(0)}(x) = (1 + O(x))\, x^{T_0}, \tag{2.18}$$

where

$$T_0 = \begin{pmatrix} \theta_0 & 0 \\ 0 & -\theta_0 \end{pmatrix}. \tag{2.19}$$

The local monodromy of $Y^{(0)}(x)$ around $x = 0$ is

$$M_0 = e^{2\pi i T_0}. \tag{2.20}$$

2) At the irregular singularity $x = \infty$, a formal solution is given by

$$Y^{(\infty)} = \left(1 + \frac{Y_1}{x} + \cdots\right) e^{T(x)}, \tag{2.21}$$

$$T(x) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{x^2}{2} + \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} x + \begin{pmatrix} \theta_\infty & 0 \\ 0 & -\theta_\infty \end{pmatrix} \log \frac{1}{x}, \tag{2.22}$$

$$Y_1 = \frac{1}{2} \begin{pmatrix} -H_{IV} & u \\ 2(z - \theta_0 - \theta_\infty)u & H_{IV} \end{pmatrix}, \tag{2.23}$$

where

$$H_{IV} = \frac{2}{y} z^2 - \left(y + 2t + \frac{4}{y}\theta_0 \right) z + (\theta_0 + \theta_\infty)(y + 2t). \tag{2.24}$$

Since $x = \infty$ is an irregular singularity, the actual asymptotic behavior of $Y(x)$ changes the form in the Stokes region of the complex x-plane:

(2.25)

$$S_j = \left\{ x \;\middle|\; \frac{\pi}{2}(j-1) - \epsilon < \arg x < \frac{\pi}{2} j + \epsilon, |x| > R, \right\}, \quad (j = 1, 2, 3, 4, 5),$$

where ϵ is sufficiently small and R is sufficiently large. We denote $Y^{(j)}$ is a holomorphic solution in S_j. According to the Stokes phenomenon, if

$$Y^{(j)} \sim Y^{(\infty)}(x) \quad \text{as} \quad x \to \infty \quad \text{in} \quad S_j, \tag{2.26}$$

then

$$Y^{(j+1)} = Y^{(j)} G_j \quad \text{and} \quad Y^{(5)} = Y^{(1)} e^{2\pi i T_0}, \tag{2.27}$$

where the matrices $G_j, (1 \le j \le 4)$ are called the Stokes matrices and $e^{2\pi i T_0}$ is a formal monodromy around $x = \infty$.

3) Connection matrix Γ

Since both $Y^{(0)}$ and $Y^{(1)}$ satisfy (2.2), they are related by the connection matrix:

$$Y^{(1)} = Y^{(0)} \Gamma. \tag{2.28}$$

4) We have

$$\Gamma^{-1} M_0 \Gamma G_1 G_2 G_3 G_4 e^{2\pi i T_0} = I_2. \tag{2.29}$$

Generally, we cannot calculate G_i and Γ. By the isomonodromy condition, the linear monodromy is invariant for any t. For the symmetric solution of the fourth Painlevé equation we can calculate the linear monodromy, because (2.2) is reduced to the Whittaker equation when $t = 0$.

Theorem 2. *For the symmetric solution* (2.9) *and* (2.10) *of the fourth Painlevé equation, the linear monodromy is*

$$M_0 = \begin{pmatrix} e^{2i\pi\theta_0} & 0 \\ 0 & e^{2i\pi(1-\theta_0)} \end{pmatrix} = \begin{pmatrix} -e^{4mi\pi} & 0 \\ 0 & -e^{-4mi\pi} \end{pmatrix}, \tag{2.30}$$

$$\Gamma = \begin{pmatrix} \frac{\Gamma(-2m)}{\Gamma\left(\frac{1}{2}-m-k\right)} & \frac{\Gamma(-2m)e^{-i\pi(k+m+\frac{1}{2})}}{\Gamma\left(\frac{1}{2}-m+k\right)} \\ \frac{\Gamma(2m)}{\Gamma\left(\frac{1}{2}+m-k\right)} & \frac{\Gamma(2m)e^{-i\pi(k-m+\frac{1}{2})}}{\Gamma\left(\frac{1}{2}+m+k\right)} \end{pmatrix}, \tag{2.31}$$

$$G_1 = \begin{pmatrix} 1 & 0 \\ \frac{2\pi e^{i\pi(\frac{-1}{2}+2k)}}{\Gamma\left(\frac{1}{2}-m-k\right)\Gamma\left(\frac{1}{2}+m-k\right)} & 1 \end{pmatrix}, \tag{2.32}$$

$$G_2 = \begin{pmatrix} 1 & \frac{2\pi e^{i\pi(\frac{-1}{2}-4k)}}{\Gamma\left(\frac{1}{2}-m+k\right)\Gamma\left(\frac{1}{2}+m+k\right)} \\ 0 & 1 \end{pmatrix}, \tag{2.33}$$

$$G_3 = \begin{pmatrix} 1 & 0 \\ \frac{2\pi e^{i\pi(\frac{-1}{2}+6k)}}{\Gamma\left(\frac{1}{2}-m-k\right)\Gamma\left(\frac{1}{2}+m-k\right)} & 1 \end{pmatrix}, \tag{2.34}$$

$$G_4 = \begin{pmatrix} 1 & \frac{2\pi e^{i\pi(\frac{-1}{2}-8k)}}{\Gamma\left(\frac{1}{2}-m+k\right)\Gamma\left(\frac{1}{2}+m+k\right)} \\ 0 & 1 \end{pmatrix}, \tag{2.35}$$

$$e^{2i\pi T_0} = \begin{pmatrix} e^{2i\pi(1-\theta_\infty)} & 0 \\ 0 & e^{2i\pi\theta_\infty} \end{pmatrix} = \begin{pmatrix} -e^{-4ki\pi} & 0 \\ 0 & -e^{4ki\pi} \end{pmatrix}. \tag{2.36}$$

For special parameters, we have

Corollary 3. *We set* $2\theta_\infty - 1 = \alpha_0 - \alpha_2$, $2\theta_0 = -\alpha_1$ *and* $\alpha_0 + \alpha_1 + \alpha_2 = 1$.

1) *In case of* $\alpha_0 = 0$, *we have* $m + k = -1/2$ *and* $G_2 = G_4 = I_2$.

2) *In case of* $\alpha_2 = 0$, *we have* $m - k = -1/2$ *and* $G_1 = G_3 = I_2$.

3) *In case of* $\alpha_0 = 0$ *and* $\alpha_2 = 0$, *we have* $G_1 = G_2 = G_3 = G_4 = I_2$.

3. Comparison with classical solutions. Umemura studied special solutions of the Painlevé equations [14]. Umemura's classical solutions are either rational solution or the Riccati solution [10, 11, 15]. We show that the symmetric solution of the fourth Painlevé equation includes rational solutions and one point of the Riccati solution of Umemura's classical solutions.

1) The Riccati solution. We set $p = y + 2t - 2w$. Then the system (2.5) and (2.6) is equivalent to the following system:

(3.1) $$\frac{dy}{dt} = 2yp - y^2 - 2ty + 4\theta_0,$$

(3.2) $$\frac{dp}{dt} = 2yp - p^2 + 2tp + 2(\theta_0 - \theta_\infty + 1).$$

If $\alpha_2 = 0$, $\theta_0 - \theta_\infty + 1 = 0$. $p = 0$ is a special solution and y satisfies the Riccati equation

(3.3) $$\frac{dy}{dt} = -y^2 - 2ty + 4\theta_0,$$

which is solved by the Weber function. In this case, the linear monodromy is upper triangular matrices by Corollary 3 (2).

If $y(0) = 0$ in (3.1), the Riccati solution is a symmetric solution. We remark that the Riccati solutions have the same linear monodromy.

2) Rational solutions.

2-1) If $\alpha_0 = \alpha_2 = 0$, $\theta_0 = -1/2$. The Riccati equation is

(3.4) $$\frac{dy}{dt} = y^2 + 2ty - 2,$$

which has a rational solution $y = -2t$.

This solution is reduced to the Hermite polynomial. $(y, w) = (-2t, 0)$ is a symmetric solution of the fourth Painlevé equation. In this case, every Stokes matrix is a unit matrix by Corollary 3 (3).

2-2) If $\alpha_0 = \alpha_1 = \alpha_2 = 1/3$, the fourth Painlevé equation has an rational solution:

(3.5) $$y = \frac{-2t}{3}, \qquad w = \frac{t}{3},$$

which is a symmetric solution of the fourth Painlevé equation. Since we have $(k, m) = (0, -1/3)$, (2.15) is reduced to the Airy function.

4. Conclusion. 1) The symmetric solution of the fourth Painlevé equation exists for any parameter α and β.

2) There exist rational solutions and the Riccati solutions for the fourth Painlevé equation for special parameters. Only for such special parameters, the symmetric solution coincides with Umemura's classical solution. In this sense, the symmetric solution is a new special solution beyond Umemura's class.

3) Two of four Stokes matrices (G_1 and G_3 or G_2 and G_4) become unit matrices when α_0 or $\alpha_2 = 0$, and every Stokes matrix becomes a unit matrix when $\alpha_0 = \alpha_2 = 0$.

Especially when $\alpha_2 = 0$, the linear monodromy become upper triangular matrices.

When $\alpha_0 = \alpha_1 = \alpha_2 = 1/3$ and $y = -2t/3$, the solution of the associated linear equation can be solved by the Airy function.

5. Appendix. In this section, the Stokes matrices are derived [4].

1) Two fundamental solutions $X_{k,m}(x)$ and $X_{-k,m}(xe^{\frac{-i\pi}{2}})$ in the Stokes region S_j are expressed in the linear combination of $L_{k,m}(x)$ and $L_{k,-m}(x)$.

For $r, s, t \in Z$,

(5.1)
$$X_{k,m}(xe^{ri\pi}) = \frac{\Gamma(-2m)e^{ri\pi\theta_0}L_{k,m}(x)}{\Gamma\left(\frac{1}{2} - m - k\right)} + \frac{\Gamma(2m)e^{ri\pi(1-\theta_0)}L_{k,-m}(x)}{\Gamma\left(\frac{1}{2} + m - k\right)},$$

(5.2)
$$X_{k,m}(xe^{si\pi}) = \frac{\Gamma(-2m)e^{si\pi\theta_0}L_{k,m}(x)}{\Gamma\left(\frac{1}{2} - m - k\right)} + \frac{\Gamma(2m)e^{si\pi(1-\theta_0)}L_{k,-m}(x)}{\Gamma\left(\frac{1}{2} + m - k\right)},$$

(5.3)
$$X_{-k,m}(xe^{ti\pi - \frac{i\pi}{2}}) = \frac{\Gamma(-2m)e^{i\pi\theta_0(t-\frac{1}{2})}L_{k,m}(x)}{\Gamma\left(\frac{1}{2} - m + k\right)} + \frac{\Gamma(2m)e^{i\pi(t-\frac{1}{2})(1-\theta_0)}L_{k,-m}(x)}{\Gamma\left(\frac{1}{2} + m + k\right)}$$

hold.

Eliminating $L_{k,m}$, $L_{k,-m}$, and putting $s = 0$, $t = 0$ and $x \to xe^{-ri\pi}$, then we have

(5.4) $$X_{k,m}(x) \sim C_r e^{\frac{-x^2}{2}} x^{\theta_\infty - 1} + D_r e^{\frac{x^2}{2}} x^{-\theta_\infty},$$
$$\left(r - \frac{1}{4}\right)\pi < \arg x < \left(r + \frac{3}{4}\right)\pi, \quad (r = 0, 1, 2, \ldots).$$

Similary, we have

(5.5)
$$X_{-k,m}(xe^{\frac{-i\pi}{2}}) \sim E_r e^{\frac{-x^2}{2}} x^{\theta_\infty - 1} + F_r e^{\frac{x^2}{2}} x^{-\theta_\infty},$$
$$\left(r - \frac{1}{4}\right)\pi < \arg x < \left(r + \frac{3}{4}\right)\pi, \quad (r = 0, 1, 2, \ldots),$$

where

(5.6)
$$C_r = e^{r(1-\theta_\infty)i\pi} e^{\frac{ri\pi}{2}} \left[\frac{\sin 2(r+1)m\pi}{\sin 2m\pi} + e^{-2ki\pi} \frac{\sin 2rm\pi}{\sin 2m\pi}\right],$$

(5.7)
$$D_r = e^{(r+\frac{1}{2})\theta_\infty i\pi} \frac{-2\pi e^{\frac{\pi}{2}i(r+1)} e^{-ki\pi} e^{-\frac{i\pi}{4}} \sin 2rm\pi}{\Gamma\left(\frac{1}{2} - m - k\right)\Gamma\left(\frac{1}{2} + m - k\right)\sin 2m\pi},$$

(5.8)

$$E_r = e^{[r(1-\theta_\infty)-\frac{\theta_\infty}{2}]i\pi} \frac{e^{\frac{\pi}{2}ir}e^{-\frac{i\pi}{4}}e^{-ki\pi}2\pi\sin 2rm\pi}{\Gamma\left(\frac{1}{2}-m+k\right)\Gamma\left(\frac{1}{2}+m+k\right)\sin 2m\pi},$$

(5.9)

$$F_r = -e^{r\theta_\infty i\pi}e^{\frac{\pi}{2}ir}\left[\frac{\sin 2(r-1)m\pi}{\sin 2m\pi}+e^{-2ki\pi}\frac{\sin 2rm\pi}{\sin 2m\pi}\right].$$

2) Stokes matrices G_j

For $r\pi < \arg x < \left(r+\frac{1}{2}\right)\pi$, $(r \in Z)$, we write the coefficient matrix of (5.4), (5.5) as

$$\begin{pmatrix} C_r & E_r \\ D_r & F_r \end{pmatrix}. \tag{5.10}$$

For $\left(r+\frac{1}{2}\right)\pi < \arg x < (r+1)\pi$, we have

$$\begin{pmatrix} C_r & E_r \\ D_{r+1} & F_{r+1} \end{pmatrix}, \tag{5.11}$$

$$G_{2r+1}\begin{pmatrix} C_r & E_r \\ D_{r+1} & F_{r+1} \end{pmatrix} = \begin{pmatrix} C_r & E_r \\ D_r & F_r \end{pmatrix}, \tag{5.12}$$

where

$$G_{2r+1} = \begin{pmatrix} 1 & 0 \\ T_{2r+1} & 1 \end{pmatrix}, \tag{5.13}$$

$$T_{2r+1} = \frac{D_r - D_{r+1}}{C_r} = \frac{F_r - F_{r+1}}{E_r}. \tag{5.14}$$

Substituting (5.6) and (5.7), we have

$$T_{2r+1} = \frac{2\pi e^{i\pi(\frac{-1}{2}+(4r+2)k)}}{\Gamma\left(\frac{1}{2}+m-k\right)\Gamma\left(\frac{1}{2}-m-k\right)}, \quad (r=0,1,2,\dots). \tag{5.15}$$

In similar way, we have

$$G_{2r} = \begin{pmatrix} 1 & T_{2r} \\ 0 & 1 \end{pmatrix}, \tag{5.16}$$

$$T_{2r} = \frac{2\pi e^{i\pi(\frac{-1}{2}-4rk)}}{\Gamma\left(\frac{1}{2}+m+k\right)\Gamma\left(\frac{1}{2}-m+k\right)}, \quad (r=1,2,\dots). \tag{5.17}$$

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