Values of absolute tensor products

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Abstract: We study values of absolute tensor products (multiple zeta functions) at integral arguments. We obtain a simple formula for the absolute value of the double sine function. We express values of the multiple gamma function related to the functional equation.

Key words: Absolute tensor product; multiple zeta function; multiple sine function; multiple gamma function.

1. Introduction. We study values of absolute tensor products (multiple zeta functions) of two types: $\zeta(s,\mathbf{F}_{p_1})\otimes\cdots\otimes\zeta(s,\mathbf{F}_{p_r})$ and the multiple gamma function $\Gamma_r(s)$.

Let p and q be prime numbers. In $\mathrm{Re}(s)>0$ we define $\zeta_{p,q}(s)$ as follows:

$$\zeta_{p,q}(s)=\begin{cases}\exp\left(-\dfrac{i}{2}\displaystyle\sum_{n=1}^{\infty}\frac{\cot(\pi n(\log p/\log q))}{n}p^{-ns}-\dfrac{i}{2}\displaystyle\sum_{n=1}^{\infty}\frac{\cot(\pi n(\log q/\log p))}{n}q^{-ns}-\dfrac{1}{2}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}p^{-ns}-\dfrac{1}{2}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}q^{-ns}\right) & \text{if } p\neq q,\\ \exp\left(\dfrac{i}{2\pi}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}p^{-ns}-\left(1-\dfrac{i\log p}{2\pi}s\right)\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}p^{-ns}\right) & \text{if } p=q.\end{cases}$$

In [7, 8], $\zeta_{p,q}(s)$ is identified with the absolute tensor product $\zeta(s,\mathbf{F}_p)\otimes\zeta(s,\mathbf{F}_q)$, where $\zeta(s,\mathbf{F}_p)=(1-p^{-s})^{-1}$ is the Hasse zeta function of the finite field $\mathbf{F}_p$ (or of the scheme $\mathrm{Spec}(\mathbf{F}_p)$). We recall the following results proved in [7, 8].

Theorem A. *Let p and q be distinct prime numbers.*

(1) *$\zeta_{p,q}(s)$ converges absolutely in* $\mathrm{Re}(s)>0$.

(2) *$\zeta_{p,q}(s)$ has an analytic continuation to all $s\in\mathbf{C}$ as a meromorphic function of order* 2.

(3) *All the zeros and poles are simple and they are located as follows:*

$$\text{zeros at } s\in\frac{2\pi i}{\log p}\mathbf{Z}_{\geq 0}+\frac{2\pi i}{\log q}\mathbf{Z}_{\geq 0},$$
$$\text{poles at } s\in\frac{2\pi i}{\log p}\mathbf{Z}_{<0}+\frac{2\pi i}{\log q}\mathbf{Z}_{<0}.$$

(4) *$\zeta_{p,q}(s)$ has the following functional equation:*

$$\begin{aligned}\zeta_{p,q}(-s)&=\zeta_{p,q}(s)^{-1}(pq)^{s/2}(1-p^{-s})(1-q^{-s})\\&\quad\times\exp\left(\frac{i\log p\log q}{4\pi}s^2-\frac{\pi i}{6}\left(\frac{\log q}{\log p}+\frac{\log p}{\log q}+3\right)\right).\end{aligned}$$

Theorem B. *Let p be a prime number.*

(1) *$\zeta_{p,p}(s)$ converges absolutely in* $\mathrm{Re}(s)>0$.

(2) *$\zeta_{p,p}(s)$ has an analytic continuation to all $s\in\mathbf{C}$ as a meromorphic function of order* 2.

(3) *$\zeta_{p,p}(s)$ has the following zeros and poles:*

$$\text{zeros at } s=\frac{2\pi in}{\log p}\ \text{for } n\in\mathbf{Z}_{\geq 0}\ \text{of order } n+1$$
$$\text{poles at } s=-\frac{2\pi in}{\log p}\ \text{for } n\in\mathbf{Z}_{\geq 2}\ \text{of order } n-1$$

(4) *$\zeta_{p,p}(s)$ has the functional equation:*

$$\zeta_{p,p}(-s)=\zeta_{p,p}(s)^{-1}p^s(1-p^{-s})^2\times\exp\left(\frac{i(\log p)^2}{4\pi}s^2-\frac{5\pi i}{6}\right).$$

We remark that Theorems A and B remind us of the simple result:

Theorem C. *Let p be a prime number, and let*

$$\zeta_p(s)=\exp\left(\sum_{n=1}^{\infty}\frac{1}{n}p^{-ns}\right)$$

in $\mathrm{Re}(s)>0$. *Then:*

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(1) $\zeta_p(s)$ *converges absolutely in* $\mathrm{Re}(s) > 0$.
(2) $\zeta_p(s)$ *has an analytic continuation to all* $s \in \mathbf{C}$ *as a meromorphic function of order* 1.
(3) $\zeta_p(s)$ *has no zeros, and* $\zeta_p(s)$ *has the poles at* $s \in (2\pi i/\log p)\mathbf{Z}$, *which are all simple.*
(4) $\zeta_p(s)$ *has the functional equation* $\zeta_p(-s) = \zeta_p(s)(-p^{-s})$.

Of course, Theorem C is easily seen from the expression

$$\zeta_p(s) = (1 - p^{-s})^{-1}.$$

On the other hand, Theorems A and B are not so easy to show and we used the theory of double sine functions developed in [2, 5, 6]. In this paper we first study the values $\zeta_{p,q}(-m)$ for integers $m \geq 1$. We obtain the following results:

Theorem 1. *Let p and q be prime numbers. Then*

$$|\zeta_{p,q}(-m)| = \sqrt{(p^m - 1)(q^m - 1)}$$

for integers $m \geq 1$.

Theorem 2.

$$\zeta_{2,2}(-1) = -e^{(\pi i)/8}.$$

Next, we study the triple case

$$\begin{aligned} &\zeta_{p,p,p}(s) \\ &= \exp\Biggl(-\frac{1}{4\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^3}p^{-ns} \\ &\quad + \frac{i}{2\pi}\left(\frac{is\log p}{2\pi} - \frac{3}{2}\right)\sum_{n=1}^{\infty}\frac{1}{n^2}p^{-ns} \\ &\quad + \frac{1}{2}\left(\frac{is\log p}{2\pi} - 1\right)\left(\frac{is\log p}{2\pi} - 2\right) \\ &\quad \times \sum_{n=1}^{\infty}\frac{1}{n}p^{-ns}\Biggr). \end{aligned}$$

We report the following concrete result:

Theorem 3.

$$|\zeta_{2,2,2}(-1)| = e^{-(7\zeta(3))/(32\pi^2)}.$$

We recall that Quillen [14] and Lichtenbaum [11] gave an excellent interpretation for

$$|\zeta_p(-m)|^{-1} = p^m - 1$$

as the order of K-group

$$|\zeta_p(-m)|^{-1} = \# K_{2m-1}(\mathbf{F}_p).$$

It would be interesting to give some interpretation such as

$$|\zeta_{p,q}(-m)| = \sqrt{\# K_{2m-1}(\mathbf{F}_p \otimes_{\mathbf{F}_1} \mathbf{F}_q)},$$

where $\mathbf{F}_1$ is the (virtual) field of one element (see [10, 12, 13, 15]).

Next, we investigate values of the multiple gamma function considered as a multiple zeta function (absolute tensor product). We recall that the multiple gamma function $\Gamma_r(s)$ of Barnes [1] is defined as

$$\begin{aligned} \Gamma_r(s) &= \left(\prod_{n_1,\ldots,n_r \geq 0}(n_1 + \cdots + n_r + s)\right)^{-1} \\ &= \exp\left(\frac{\partial}{\partial w}\zeta_r(w,s)\Big|_{w=0}\right) \end{aligned}$$

where

$$\zeta_r(w,s) = \sum_{n_1,\ldots,n_r \geq 0}(n_1 + \cdots + n_r + s)^{-w}$$

is the multiple Hurwitz zeta function. We know that

$$\Gamma_1(s) = \frac{\Gamma(s)}{\sqrt{2\pi}}$$

from Lerch's formula (1894) for the usual gamma function $\Gamma(s)$. As explained by Manin [12, p. 134] $\Gamma_1(s)$ is viewed as the zeta function of the "dual infinite dimensional projective space over $\mathbf{F}_1$" $\check{\mathbf{P}}^\infty(\mathbf{F}_1)$:

$$\Gamma_1(s) = \zeta(s, \check{\mathbf{P}}^\infty(\mathbf{F}_1)) = \left(\prod_{n \geq 0}(s+n)\right)^{-1}.$$

Then, $\Gamma_r(s)$ is obtained as an absolute tensor product:

$$\Gamma_r(s) = (\Gamma_1(s)^{\otimes r})^{(-1)^{r-1}}.$$

Hence, from the viewpoint of Quillen [14] and Lichtenbaum [11], it would be interesting to study values at $s = -n$ $(n \geq 0)$ of $\Gamma_r(s)$.

It turns out that $\Gamma_r(s)$ has a functional equation $s \leftrightarrow r - s$, and that the value at $s = -n$ is intimately related to the value at $s = r + n$. Here we describe the cases $r = 1$ and 2. (We refer to [9] for a more general treatment.)

Theorem 4. *Let* $n \geq 0$ *be an integer. Then*:
(1) $\Gamma_1(s)(s+n)|_{s=-n} = (-1)^n/(n!\sqrt{2\pi})$.
(2) $\Gamma_1(1+n) = n!/\sqrt{2\pi}$.

Theorem 5. *Let $n \geq 0$ be an integer. Then*:

(1)
$$\Gamma_2(s)(s+n)^{n+1}|_{s=-n} = (-1)^{(n(n+1))/2} e^{\zeta'(-1)} \times (2\pi)^{-(n+1)/2}(1!\,2!\cdots n!)^{-1}.$$

(2) $\Gamma_2(2+n) = e^{\zeta'(-1)}(2\pi)^{(n+1)/2}(1!\,2!\cdots n!)^{-1}$.

It is easy to identify values in Theorems 4 and 5 by $\#GL_n(\mathbf{F}_1) = n! = \#S_n$ and it might be possible as in [4] to interpret these values via the order of a suitable K-group of $\check{\mathbf{P}}^{\infty}(\mathbf{F}_1)^{\otimes r}$ for $r = 1$ and 2; we notice that Manin [12] and Soulé [15] identify the K-group $K_m(\mathbf{F}_1)$ as the stable homotopy group π_m^s.

We remark that relations between (1) and (2) in Theorems 4 and 5 are indicating the functional equation $\Gamma_r(s) \leftrightarrow \Gamma_r(r-s)$ and they are reformulated using the derivatives of the multiple sine function

$$S_r(s) = \Gamma_r(s)^{-1}\Gamma_r(r-s)^{(-1)^r}$$

(see [5] for the general theory) as follows:

Theorem 6. *Let $n \geq 0$ be an integer. Then*:

(1) $(S_1(s)/(s+n))|_{s=-n} = (-1)^n 2\pi$.

(2) $(S_2(s)/(s+n)^{n+1})|_{s=-n} = (-1)^{(n(n+1))/2} (2\pi)^{n+1}$.

It would be valuable to report simple facts about absolute tensor products of basic zeta functions over $\mathbf{F}_1$. Zeta functions of the affine space and the projective space over $\mathbf{F}_1$ are given as

$$\zeta\left(s, \mathbf{A}^k_{\mathbf{F}_1}\right) = \frac{1}{s-k}$$

and

$$\zeta\left(s, \mathbf{P}^k_{\mathbf{F}_1}\right) = \frac{1}{s(s-1)\cdots(s-k)}.$$

Hence the corresponding absolute tensor products are

$$\zeta\left(s, \mathbf{A}^{k_1}_{\mathbf{F}_1}\right) \otimes \cdots \otimes \zeta\left(s, \mathbf{A}^{k_r}_{\mathbf{F}_1}\right) = (s-(k_1+\cdots+k_r))^{(-1)^r}$$

and

$$\zeta\left(s, \mathbf{P}^{k_1}_{\mathbf{F}_1}\right) \otimes \cdots \otimes \zeta\left(s, \mathbf{P}^{k_r}_{\mathbf{F}_1}\right) = \prod_{\substack{j_i=0,\dots,k_i \\ (i=1,\dots,r)}} (s-(j_1+\cdots+j_r))^{(-1)^r}.$$

We refer to [4] for further study.

2. $\boldsymbol{\zeta(s,\mathbf{F}_{p_1}) \otimes \cdots \otimes \zeta(s,\mathbf{F}_{p_r})}$.

Proof of Theorem 1.

(1) $p \neq q$ case: From the functional equation (Theorem A (4)) for $\zeta_{p,q}(s)$ we have

$$\begin{aligned}\zeta_{p,q}(-m) &= \zeta_{p,q}(m)^{-1}(pq)^{m/2}(1-p^{-m})(1-q^{-m}) \\ &\quad \times \exp\left(\frac{i(\log p)(\log q)}{4\pi}m^2 - \frac{\pi i}{6}\left(\frac{\log q}{\log p} + \frac{\log p}{\log q} + 3\right)\right)\end{aligned}$$

with

$$\begin{aligned}\zeta_{p,q}(m) = \exp\Bigg(&-\frac{i}{2}\sum_{n=1}^{\infty}\frac{\cot(\pi n(\log p/\log q))}{n}p^{-nm} \\ &-\frac{i}{2}\sum_{n=1}^{\infty}\frac{\cot(\pi n(\log q/\log p))}{n}q^{-nm} \\ &-\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}p^{-nm} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}q^{-nm}\Bigg).\end{aligned}$$

Hence we get

$$\begin{aligned}\zeta_{p,q}(-m) &= \sqrt{(p^m-1)(q^m-1)} \\ &\quad \times \exp\Bigg(\frac{i}{2}\sum_{n=1}^{\infty}\frac{\cot(\pi n(\log p/\log q))}{n}p^{-nm} \\ &\qquad + \frac{i}{2}\sum_{n=1}^{\infty}\frac{\cot(\pi n(\log q/\log p))}{n}q^{-nm} \\ &\qquad + \frac{i(\log p)(\log q)}{4\pi}m^2 \\ &\qquad - \frac{\pi i}{6}\left(\frac{\log q}{\log p} + \frac{\log p}{\log q} + 3\right)\Bigg).\end{aligned}$$

Thus

$$|\zeta_{p,q}(-m)| = \sqrt{(p^m-1)(q^m-1)}.$$

(2) $p = q$ case: The functional equation (Theorem B (4)) implies that

$$\zeta_{p,p}(-m) = \zeta_{p,p}(m)^{-1}p^m(1-p^{-m})^2 \times \exp\left(\frac{i(\log p)^2}{4\pi}m^2 - \frac{5\pi i}{6}\right)$$

with

$$\zeta_{p,p}(m) = \exp\left(\frac{i}{2\pi}\mathrm{Li}_2(p^{-m}) - \left(1 - \frac{i\log p}{2\pi}m\right)\mathrm{Li}_1(p^{-m})\right)$$

$$= (1 - p^{-m}) \exp\biggl(\frac{i}{2\pi} \mathrm{Li}_2(p^{-m}) + \frac{i \log p}{2\pi} m \mathrm{Li}_1(p^{-m}) \biggr),$$

where we use the polylogarithm

$$\mathrm{Li}_r(u) = \sum_{n=1}^{\infty} \frac{u^n}{n^r}.$$

Hence

$$\zeta_{p,p}(-m) = (p^m - 1) \times \exp\biggl(-\frac{i}{2\pi} \mathrm{Li}_2(p^{-m}) - \frac{i(\log p)^2}{4\pi} m^2 + \frac{i \log p}{2\pi} m \log(p^m - 1) - \frac{5\pi i}{6} \biggr)$$

gives

$$|\zeta_{p,p}(-m)| = p^m - 1.$$

□

Proof of Theorem 2. The calculation in the above proof of Theorem 1 shows that

$$\zeta_{2,2}(-1) = \exp\left(-\frac{i}{2\pi} \mathrm{Li}_2\left(\frac{1}{2}\right) - \frac{i(\log 2)^2}{4\pi} - \frac{5\pi i}{6} \right).$$

Hence, using Euler's result

$$\mathrm{Li}_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{1}{2}(\log 2)^2$$

we have

$$\begin{aligned} \zeta_{2,2}(-1) &= \exp\left(-\frac{7\pi}{8} i \right) \\ &= -e^{(\pi/8)i} \\ &= -\frac{\sqrt{2+\sqrt{2}}}{2} - i\frac{\sqrt{2-\sqrt{2}}}{2}. \end{aligned}$$

□

Proof of Theorem 3. From the explicit expression for the triple sine function $S_3(x)$ shown in [5] using another multiple sine function $\mathcal{S}_r(x)$ we get

$$S_3(x) = e^{\zeta(3)/4\pi^2} \mathcal{S}_3(x)^{1/2} \mathcal{S}_2(x)^{-3/2} \mathcal{S}_1(x)$$

and

$$\zeta_{p,p,p}(s) = S_3\left(\frac{is \log p}{2\pi} \right)^{-1} \times \exp\left(-\frac{(\log p)^3}{48\pi^2} s^3 - i\frac{3(\log p)^2}{16\pi} s^2 + \frac{\log p}{2} s + i\frac{\pi}{2} \right)$$

in $\mathrm{Re}(s) > 0$ at first. Since the both sides are meromorphic functions on all $s \in \mathbf{C}$, this identity holds for all $s \in \mathbf{C}$. In particular

$$\zeta_{p,p,p}(-s) = S_3\left(-\frac{is \log p}{2\pi} \right)^{-1} \times \exp\left(\frac{(\log p)^3}{48\pi^2} s^3 - i\frac{3(\log p)^2}{16\pi} s^2 - \frac{\log p}{2} s + i\frac{\pi}{2} \right).$$

Hence, using

$$\begin{aligned} S_3(-x) &= e^{\zeta(3)/4\pi^2} \mathcal{S}_3(-x)^{1/2} \mathcal{S}_2(-x)^{-3/2} \mathcal{S}_1(-x) \\ &= -e^{\zeta(3)/4\pi^2} \mathcal{S}_3(x)^{1/2} \mathcal{S}_2(x)^{3/2} \mathcal{S}_1(x) \\ &= -S_3(x) \mathcal{S}_2(x)^3 \end{aligned}$$

we have

$$\begin{aligned} \zeta_{p,p,p}(-s) &= -S_3\left(\frac{is \log p}{2\pi} \right)^{-1} \mathcal{S}_2\left(\frac{is \log p}{2\pi} \right)^{-3} \\ &\quad \times \exp\left(\frac{(\log p)^3}{48\pi^2} s^3 - i\frac{3(\log p)^2}{16\pi} s^2 - \frac{\log p}{2} s + i\frac{\pi}{2} \right) \\ &= -\zeta_{p,p,p}(s) \mathcal{S}_2\left(\frac{is \log p}{2\pi} \right)^{-3} \\ &\quad \times \exp\left(\frac{(\log p)^3}{24\pi^2} s^3 - (\log p)s \right). \end{aligned}$$

In particular

$$\zeta_{p,p,p}(-m) = -\zeta_{p,p,p}(m) \mathcal{S}_2\left(\frac{im \log p}{2\pi} \right)^{-3} \times \exp\left(\frac{(\log p)^3}{24\pi^2} m^3 - (\log p)m \right).$$

Since

$$|\mathcal{S}_2(x)| = 1 \text{ for } x \in i\mathbf{R},$$

we see that

$$|\zeta_{p,p,p}(-m)| = |\zeta_{p,p,p}(m)| \exp\left(\frac{(\log p)^3}{24\pi^2} m^3 - (\log p)m \right).$$

Hence

$$|\zeta_{2,2,2}(-1)| = |\zeta_{2,2,2}(1)| \exp\left(\frac{(\log 2)^3}{24\pi^2} - \log 2 \right)$$

with

$$
\begin{aligned}
&|\zeta_{2,2,2}(1)| \\
&= \exp\left(-\frac{1}{4\pi^2}\mathrm{Li}_3\left(\frac{1}{2}\right) - \frac{\log 2}{4\pi^2}\mathrm{Li}_2\left(\frac{1}{2}\right)\right. \\
&\qquad \left. + \left(-\frac{(\log 2)^2}{8\pi^2} + 1\right)\mathrm{Li}_1\left(\frac{1}{2}\right)\right).
\end{aligned}
$$

Thus, from

$$
\begin{aligned}
\mathrm{Li}_1\left(\frac{1}{2}\right) &= \log 2, \\
\mathrm{Li}_2\left(\frac{1}{2}\right) &= \frac{\pi^2}{12} - \frac{1}{2}(\log 2)^2, \\
\mathrm{Li}_3\left(\frac{1}{2}\right) &= \frac{7}{8}\zeta(3) - \frac{\pi^2}{12}\log 2 + \frac{(\log 2)^3}{6},
\end{aligned}
$$

we have

$$
|\zeta_{2,2,2}(1)| = \exp\left(-\frac{7\zeta(3)}{32\pi^2} - \frac{(\log 2)^3}{24\pi^2} + \log 2\right)
$$

and consequently

$$
|\zeta_{2,2,2}(-1)| = \exp\left(-\frac{7\zeta(3)}{32\pi^2}\right).
$$

□

3. $\Gamma_r(s)$.

Proof of Theorem 4.

(1) Notice that

$$
\begin{aligned}
\Gamma_1(s)(s+n) &= \frac{1}{\sqrt{2\pi}}\Gamma(s)(s+n) \\
&= \frac{1}{\sqrt{2\pi}}\frac{\Gamma(s)s(s+1)\cdots(s+n)}{s(s+1)\cdots(s+n-1)} \\
&= \frac{1}{\sqrt{2\pi}}\frac{\Gamma(s+n+1)}{s(s+1)\cdots(s+n-1)}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\Gamma_1(s)(s+n)|_{s=-n} &= \frac{1}{\sqrt{2\pi}}\frac{\Gamma(1)}{(-n)(-n+1)\cdots(-1)} \\
&= (-1)^n\frac{1}{\sqrt{2\pi}}\cdot\frac{1}{n!}.
\end{aligned}
$$

(2) This is easily seen from $\Gamma_1(s) = \Gamma(s)/\sqrt{2\pi}$ and the well-known formula $\Gamma(n+1) = n!$.

□

Proof of Theorem 5.

(1) From

$$
\begin{aligned}
\Gamma_2(s) &= \Gamma_2(s+1)\Gamma_1(s) \\
&= \Gamma_2(s+2)\Gamma_1(s)\Gamma_1(s+1) \\
&= \cdots \\
&= \Gamma_2(s+n+1)\Gamma_1(s)\Gamma_1(s+1)\cdots\Gamma_1(s+n)
\end{aligned}
$$

we have

$$
\begin{aligned}
&\Gamma_2(s)(s+n)^{n+1}|_{s=-n} \\
&= \Gamma_2(s+n+1)(\Gamma_1(s)(s+n))(\Gamma_1(s+1)(s+n)) \\
&\quad \cdots(\Gamma_1(s+n)(s+n))|_{s=-n} \\
&= \Gamma_2(1)\left(\frac{(-1)^n}{\sqrt{2\pi}}\frac{1}{n!}\right)\left(\frac{(-1)^{n-1}}{\sqrt{2\pi}}\frac{1}{(n-1)!}\right) \\
&\quad \cdots\left(\frac{(-1)^0}{\sqrt{2\pi}}\frac{1}{0!}\right) \\
&= e^{\zeta'(-1)}(-1)^{(n(n+1))/2}(2\pi)^{-(n+1)/2} \\
&\quad \times (1!\,2!\cdots n!)^{-1},
\end{aligned}
$$

where we used the fact

$$
\Gamma_2(1) = \exp(\zeta_2'(0,1)) = \exp(\zeta'(-1)).
$$

(2) Using the periodicity $\Gamma_2(s+1) = \Gamma_2(s)\Gamma_1(s)^{-1}$ we get

$$
\begin{aligned}
\Gamma_2(2+n) &= \Gamma_2(1+n)\Gamma_1(1+n)^{-1} \\
\Gamma_2(1+n) &= \Gamma_2(n)\Gamma_1(n)^{-1} \\
&\cdots \\
\Gamma_2(2) &= \Gamma_2(1)\Gamma_1(1)^{-1}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\Gamma_2(2+n) &= \Gamma_2(1)(\Gamma_1(1)\Gamma_1(2)\cdots\Gamma_1(1+n))^{-1} \\
&= e^{\zeta'(-1)}(2\pi)^{(n+1)/2}(1!\,2!\cdots n!)^{-1}.
\end{aligned}
$$

□

Proof of Theorem 6.

(1) Since $S_1(s) = 2\sin(\pi s)$, it is easy to see that

$$
S_1'(-n) = (-1)^n 2\pi.
$$

(2) We show that

$$
\begin{aligned}
&S_2(s) \\
&= (-1)^{(n(n+1))/2}(2\pi)^{n+1}(s+n)^{n+1} \\
&\quad + O\left((s+n)^{n+2}\right)
\end{aligned}
$$

as $s \to -n$. First, the case $n = 0$ is equivalent to $S_2'(0) = 2\pi$, which is proved in [3]. Next, for a general $n \geq 1$ we get

$$
S_2(s-n) = (-1)^{(n(n+1))/2}S_2(s)S_1(s)^n
$$

from iterating the process

$$
S_2(s-1) = S_2(s)S_1(s-1) = -S_2(s)S_1(s).
$$

Hence

$$
\begin{aligned}
&S_2(s-n) \\
&= (-1)^{(n(n+1))/2}\left(2\pi s + O\left(s^2\right)\right)\left(2\pi s + O\left(s^2\right)\right)^n \\
&= (-1)^{(n(n+1))/2}(2\pi)^{n+1}s^{n+1} + O\left(s^{n+2}\right)
\end{aligned}
$$

as $s \to 0$. Thus

$$S_2(s) = (-1)^{(n(n+1))/2}(2\pi)^{n+1}(s+n)^{n+1} + O((s+n)^{n+2})$$

as $s \to -n$. Consequently we obtain the desired

$$S_2^{(n+1)}(-n) = (-1)^{(n(n+1))/2}(2\pi)^{n+1}(n+1)!.$$

□

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