Zeta functions over $\mathbf{F}_1$

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Abstract: We show basic properties of zeta functions over the one element field starting from an algebraic set over the integer ring. We calculate several examples and we investigate special values via the associated K-group identified as the stable homotopy group of spheres.

Key words: Zeta function; one element field; K-group; stable homotopy group.

1. Introduction. The study of zeta functions over $\mathbf{F}_1$ was started by Manin [9] about 10 years ago (see [10] for a recent survey). Recently Soulé [15] and Deitmar [2] made further researches. In this paper we extend these investigations with presenting reformulations. Concerning absolute mathematics we refer to [3–7] also.

Let X be an algebraic set over $\mathbf{Z}$. We denote by

$$\zeta(s, X/\mathbf{Z}) = \prod_{p:\,\text{prime}} \zeta(s, X/\mathbf{F}_p)$$

the Hasse zeta function, where

$$\zeta(s, X/\mathbf{F}_p) = \exp\left(\sum_{m=1}^{\infty} \frac{\#X(\mathbf{F}_{p^m})}{m} p^{-ms}\right)$$

is the congruence zeta function at p.

Definition 1. We say that X is of $\mathbf{F}_1$-type if $\zeta(s, X/\mathbf{Z})$ is expressed via the Riemann zeta function $\zeta(s)$ in the form

$$\zeta(s, X/\mathbf{Z}) = \prod_{k=0}^{n} \zeta(s-k)^{a_k}$$

with $a_k \in \mathbf{Z}$.

This definition can be reformulated as follows:

Theorem 1. *Let X be an algebraic set over $\mathbf{Z}$. Then the following are equivalent.*

(1) *X is of $\mathbf{F}_1$-type by*

$$\zeta(s, X/\mathbf{Z}) = \prod_{k=0}^{n} \zeta(s-k)^{a_k}$$

with $a_k \in \mathbf{Z}$.

(2)

$$\zeta(s, X/\mathbf{F}_p) = \prod_{k=0}^{n} (1-p^{k-s})^{-a_k}$$

with $a_k \in \mathbf{Z}$ for all primes p.

(3) *There exists a polynomial*

$$N_X(t) = \sum_{k=0}^{n} a_k t^k$$

satisfying

$$\#X(\mathbf{F}_{p^m}) = N_X(p^m)$$

for all p and m.

Remark 1. Soulé [15] and Deitmar [2] used (3) to characterize a scheme X over $\mathbf{Z}$ coming from a scheme over $\mathbf{F}_1$.

Definition 2. Let X be an $\mathbf{F}_1$-type algebraic set over $\mathbf{Z}$. We define the zeta function

$$\zeta(s, X/\mathbf{F}_1) = \prod_{k=0}^{n} (s-k)^{-a_k}$$

and the Euler characteristic

$$\#X(\mathbf{F}_1) = \sum_{k=0}^{n} a_k.$$

Remark 2. We explain the reason why $\#X(\mathbf{F}_1)$ is considered to be the Euler characteristic. According to Weil's conjecture we have

$$\zeta(s, X/\mathbf{F}_p) = \prod_{l=0}^{m} P_l(p^{-s})^{(-1)^{l+1}}$$

with

$$P_l(u) = \prod_{j=1}^{b_l} (1-\alpha_{l,j}u)$$

satisfying $|\alpha_{l,j}| = p^{l/2}$, where b_l is the l-th Betti number. Hence, comparing with the expression

$$\zeta(s, X/\mathbf{F}_p) = \prod_{k=0}^{n} (1-p^{k-s})^{-a_k},$$

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we obtain

$$b_l = \begin{cases} a_{l/2} & \text{if } l \text{ is even,} \\ 0 & \text{if } l \text{ is odd.} \end{cases}$$

Thus

$$\sum_{k=0}^{n} a_k = \sum_{l=0}^{m} (-1)^l b_l$$

is the Euler characteristic.

Theorem 2. *Let X be an $\mathbf{F}_1$-type algebraic set over $\mathbf{Z}$. Then, we have the equality*

$$\zeta(s, X/\mathbf{F}_1) = \lim_{p\to 1} \zeta(s, X/\mathbf{F}_p)(p-1)^{\#X(\mathbf{F}_1)},$$

where we consider p as a complex variable when taking the limit $p \to 1$. In other words, $\zeta(s, X/\mathbf{F}_1)$ is the "leading coefficient of the Laurent expansion of $\zeta(s, X/\mathbf{F}_p)$ around $p = 1$."

Remark 3. Soulé [15] and Deitmar [2] use $\zeta(s, X/\mathbf{F}_1)^{-1}$ as the zeta function of X over $\mathbf{F}_1$. But, from Theorem 2, our definition seems to be more natural.

Theorem 3. *Basic algebraic sets $\mathbf{A}^n$, $\mathbf{P}^n$, $\mathbf{GL}_n$ and $\mathbf{SL}_n$ are $\mathbf{F}_1$-type algebraic sets over $\mathbf{Z}$. Their zeta functions have the following properties.*

(1)

$$\zeta(s, \mathbf{A}^n/\mathbf{Z}) = \zeta(s-n),$$
$$\zeta(s, \mathbf{A}^n/\mathbf{F}_p) = \frac{1}{1-p^{n-s}},$$
$$\zeta(s, \mathbf{A}^n/\mathbf{F}_1) = \frac{1}{s-n}.$$

(2)

$$\zeta(s,\mathbf{P}^n/\mathbf{Z}) = \zeta(s)\zeta(s-1)\cdots\zeta(s-n),$$
$$\zeta(s,\mathbf{P}^n/\mathbf{F}_p) = \frac{1}{(1-p^{-s})(1-p^{1-s})\cdots(1-p^{n-s})},$$
$$\zeta(s,\mathbf{P}^n/\mathbf{F}_1) = \frac{1}{s(s-1)\cdots(s-n)}.$$

(3)

$$\zeta(s, \mathbf{GL}_n/\mathbf{Z}) = \prod_{k=n(n-1)/2}^{n^2} \zeta(s-k)^{a(n,k)},$$
$$\zeta(s, \mathbf{GL}_n/\mathbf{F}_p) = \prod_{k=n(n-1)/2}^{n^2} (1-p^{k-s})^{-a(n,k)},$$
$$\zeta(s, \mathbf{GL}_n/\mathbf{F}_1) = \prod_{k=n(n-1)/2}^{n^2} (s-k)^{-a(n,k)},$$

where

$$\sum_k a(n,k)t^k = t^{(n(n-1))/2}(t-1)(t^2-1)\cdots(t^n-1).$$

(4)

$$\zeta(s, \mathbf{SL}_n/\mathbf{Z}) = \prod_{k=n(n-1)/2}^{n^2-1} \zeta(s-k)^{b(n,k)},$$
$$\zeta(s, \mathbf{SL}_n/\mathbf{F}_p) = \prod_{k=n(n-1)/2}^{n^2-1} (1-p^{k-s})^{-b(n,k)},$$
$$\zeta(s, \mathbf{SL}_n/\mathbf{F}_1) = \prod_{k=n(n-1)/2}^{n^2-1} (s-k)^{-b(n,k)},$$

where

$$\sum_k b(n,k)t^k = t^{(n(n-1))/2}(t^2-1)(t^3-1)\cdots(t^n-1).$$

(5) $\zeta(s, \mathbf{P}^n/R) = \prod_{k=0}^{n} \zeta(s, \mathbf{A}^k/R)$ *for* $R = \mathbf{Z}, \mathbf{F}_p, \mathbf{F}_1$.
(6) $\zeta(s, \mathbf{A}^n/R) = \zeta(s, \mathbf{P}^n/R)\zeta(s, \mathbf{P}^{n-1}/R)^{-1}$ *for* $R = \mathbf{Z}, \mathbf{F}_p, \mathbf{F}_1$.
(7) $\zeta(s, \mathbf{GL}_n/R) = \zeta(s-1, \mathbf{SL}_n/R)\zeta(s, \mathbf{SL}_n/R)^{-1}$ *for* $R = \mathbf{Z}, \mathbf{F}_p, \mathbf{F}_1$.
(8) $\zeta(s, \mathbf{SL}_n/R) = \prod_{k=1}^{\infty} \zeta(s+k, \mathbf{GL}_n/R)$ *for* $R = \mathbf{Z}, \mathbf{F}_p, \mathbf{F}_1$ *except for the case* $n = 1$ *and* $R = \mathbf{F}_1$ *where this equality is not valid.*

Theorem 4. *Let X be an $\mathbf{F}_1$-type algebraic set over $\mathbf{Z}$. Then*

$$\lim_{|s|\to\infty} \zeta(s, X/\mathbf{F}_1) = \begin{cases} 0 & \text{if } \#X(\mathbf{F}_1) > 0, \\ 1 & \text{if } \#X(\mathbf{F}_1) = 0, \\ \infty & \text{if } \#X(\mathbf{F}_1) < 0. \end{cases}$$

Remark 4. In contrast with the case $R = \mathbf{F}_1$ we have classically

$$\lim_{\mathrm{Re}(s)\to+\infty} \zeta(s, X/R) = 1.$$

for $R = \mathbf{Z}$ and $\mathbf{F}_p$.

Example.
(1) $\#\mathbf{A}^n(\mathbf{F}_1) = 1$ and $\lim_{|s|\to\infty} \zeta(s, \mathbf{A}^n/\mathbf{F}_1) = 0$.
(2) $\#\mathbf{P}^n(\mathbf{F}_1) = n+1$ and $\lim_{|s|\to\infty} \zeta(s, \mathbf{P}^n/\mathbf{F}_1) = 0$.
(3) $\#\mathbf{GL}_n(\mathbf{F}_1) = 0$ and $\lim_{|s|\to\infty} \zeta(s, \mathbf{GL}_n/\mathbf{F}_1) = 1$.
(4) $\#\mathbf{SL}_n(\mathbf{F}_1) = \begin{cases} 0 & (n \geq 2) \\ 1 & (n = 1) \end{cases}$ and $\lim_{|s|\to\infty} \zeta(s, \mathbf{SL}_n/\mathbf{F}_1) = \begin{cases} 1 & (n \geq 2) \\ 0 & (n = 1) \end{cases}$.

Now we look at values of $\zeta(s, X/R)$. To simplify the notation we write

$$\zeta(s, R) = \zeta(s, \mathbf{A}^0/R)$$

for $R = \mathbf{Z}, \mathbf{F}_p, \mathbf{F}_1$:

$$\zeta(s, \mathbf{Z}) = \zeta(s), \quad \text{the Riemann zeta function,}$$
$$\zeta(s, \mathbf{F}_p) = (1 - p^{-s})^{-1},$$
$$\zeta(s, \mathbf{F}_1) = \frac{1}{s}.$$

Concerning $\zeta(s, \mathbf{F}_p)$, Lichtenbaum [8] and Quillen [11] gave the interpretation:

$$|\zeta(-m, \mathbf{F}_p)| = \frac{\#K_{2m-2}(\mathbf{F}_p)}{\#K_{2m-1}(\mathbf{F}_p)}$$

for integers $m \geq 1$, where $K_n(R)$ is the (higher Quillen) K-group of R. Actually, $\#K_{2m-1}(\mathbf{F}_p) = p^m - 1$ and $\#K_{2m-2}(\mathbf{F}_p) = 1$. In the case of $\zeta(s, \mathbf{Z})$, Lichtenbaum [8] made the following

Lichtenbaum conjecture.

$$|\zeta(1 - m, \mathbf{Z})| \cong \frac{\#K_{2m-2}(\mathbf{Z})}{\#K_{2m-1}(\mathbf{Z})},$$

where $\cong$ *is indicating* "*up to some basic factors*" (*we may take* "*leading coefficients*" *also*).

It is easy to examine the value $\zeta(-m, X/R)$ for an $\mathbf{F}_1$-type algebraic set X and an integer $m \geq 1$, where $R = \mathbf{F}_1$, $\mathbf{F}_p$ and $\mathbf{Z}$. In fact, expressions

$$\zeta(s, X/\mathbf{F}_1) = \prod_{k=0}^{n} (s - k)^{-a_k}$$

and

$$\zeta(s, X/\mathbf{F}_p) = \prod_{k=0}^{n} (1 - p^{k-s})^{-a_k}$$

with $a_k \in \mathbf{Z}$ imply

$$\zeta(-m, X/\mathbf{F}_1) = (-1)^{\#X(\mathbf{F}_1)} \prod_{k=0}^{n} (m + k)^{-a_k} \in \mathbf{Q}$$

and

$$\begin{aligned}\zeta(-m, X/\mathbf{F}_p) &= (-1)^{\#X(\mathbf{F}_1)} \prod_{k=0}^{n} (p^{m+k} - 1)^{-a_k} \\ &= (-1)^{\#X(\mathbf{F}_1)} \prod_{k=0}^{n} (\#K_{2m+2k-1}(\mathbf{F}_p))^{-a_k}.\end{aligned}$$

Moreover, from

$$\zeta(s, X/\mathbf{Z}) = \prod_{k=0}^{n} \zeta(s - k)^{a_k}$$

we see that

$$\begin{aligned}\zeta(-m, X/\mathbf{Z}) &= \prod_{k=0}^{n} \zeta(-m - k)^{a_k} \\ &\cong \prod_{k=0}^{n} \left(\frac{\#K_{2m+2k}(\mathbf{Z})}{\#K_{2m+2k+1}(\mathbf{Z})} \right)^{a_k}\end{aligned}$$

under the Lichtenbaum conjecture. Thus we obtain the following result:

Theorem 5. *Let X be an $\mathbf{F}_1$-type algebraic set over $\mathbf{Z}$ with an integer $m \geq 1$.*

(1) $\zeta(-m, X/\mathbf{F}_1) \in \mathbf{Q}$.

(2) $\zeta(-m, X/\mathbf{F}_p) \in \mathbf{Q}$ *and it is written explicitly by the K-groups $K_n(\mathbf{F}_p)$.*

(3) $\zeta(-m, X/\mathbf{Z}) \in \mathbf{Q}$ *and it can be essentially written by the K-groups $K_n(\mathbf{Z})$ under the Lichtenbaum conjecture.*

To investigate $\zeta(-m, X/\mathbf{F}_1)$ K-theoretically we must know "$K_n(X/\mathbf{F}_1)$." We do not know the general definition, but we have a definition of $K_n(\mathbf{F}_1)$.

Definition 3. We define the K-group $K_n(\mathbf{F}_1)$ as the stable homotopy group π_n^s of spheres:

$$K_n(\mathbf{F}_1) = \pi_n^s = \lim_{k \to \infty} \pi_{n+k}(S^k),$$

which is stable in the range $k > n + 1$.

We notice the following points. First, this definition is given by Manin [9] and followed by Soulé [15]. Second, the naturality of the definition comes from Quillen's construction

$$K_n(R) = \pi_n(BGL_\infty(R)^+)$$

for a ring R considering $GL_\infty(\mathbf{F}_1)$ as the symmetric group S_∞.

The following result gives a partial support to a K-theoretical interpretation of values $\zeta(-m, X/\mathbf{F}_1)$.

Theorem 6. *Let m and n be non-negative integers. Then*

$$\zeta(-m, \mathbf{A}^n/\mathbf{F}_1)^{-1} \mid \#K_{2m+2n-3}(\mathbf{F}_1)$$

when $m + n$ is a prime number.

If $n = 0$ we have the following examples:

Table I. Examples

s p	$\zeta(-p, \mathbf{F}_1)^{-1}$	$\#K_{2p-3}(\mathbf{F}_1)$
2	-2	2
3	-3	$24 = 3 \cdot 2^3$
5	-5	$240 = 5 \cdot 3 \cdot 2^4$
7	-7	$504 = 7 \cdot 3^2 \cdot 2^3$
11	-11	$528 = 11 \cdot 3 \cdot 2^4$
13	-13	$1048320 = 13 \cdot 7 \cdot 5 \cdot 3^2 \cdot 2^8$

From this, it may be interesting to determine whether the equivalence

$$m \text{ is prime } \iff \zeta(-m, \mathbf{F}_1)^{-1} \mid \#K_{2m-3}(\mathbf{F}_1)$$

holds.

2. Characterization of zeta functions. We prove Theorems 1 and 2.

Proof of Theorem 1. The equivalence (1) $\iff$ (2) follows from the uniqueness of the Euler factor. To see (2) $\iff$ (3) compare

$$\log \zeta(s, X/\mathbf{F}_p) = \sum_{m=1}^{\infty} \frac{\#X(\mathbf{F}_{p^m})}{m} p^{-ms}$$

and

$$\log\left(\prod_{k=0}^{n} (1-p^{k-s})^{-a_k}\right) = \sum_{m=1}^{\infty} \frac{\sum_{k=0}^{n} a_k p^{mk}}{m} p^{-ms}.$$

□

Proof of Theorem 2. From

$$\zeta(s, X/\mathbf{F}_p)(p-1)^{\#X(\mathbf{F}_1)} = \prod_{k=0}^{n} \left(\frac{1-p^{k-s}}{p-1}\right)^{-a_k}$$

we have

$$\begin{aligned} &\lim_{p\to 1} \zeta(s, X/\mathbf{F}_p)(p-1)^{\#X(\mathbf{F}_1)} \\ &= \prod_{k=0}^{n} \left(\lim_{p\to 1} \frac{1-p^{k-s}}{p-1}\right)^{-a_k} \\ &= \prod_{k=0}^{n} (s-k)^{-a_k} \\ &= \zeta(s, X/\mathbf{F}_1). \end{aligned}$$

□

3. Examples of zeta functions.

Proof of Theorem 3. Expressions (1)–(4) follow from the following formulas for the number of rational points over a finite field $\mathbf{F}_q$:

$$\#\mathbf{A}^n(\mathbf{F}_q) = q^n,$$

$$\#\mathbf{P}^n(\mathbf{F}_q) = \sum_{k=0}^{n} q^k,$$

$$\#\mathbf{GL}_n(\mathbf{F}_q) = q^{(n(n-1))/2}(q-1)(q^2-1)\cdots(q^n-1),$$

and

$$\#\mathbf{SL}_n(\mathbf{F}_q) = q^{(n(n-1))/2}(q^2-1)(q^3-1)\cdots(q^n-1).$$

It is easy to see (5) and (6) from (1) and (2). To show (7) we use the relation

$$a(n,k) = b(n,k-1) - b(n,k)$$

coming from

$$\sum_k a(n,k)t^k = (t-1)\sum_k b(n,k)t^k.$$

We see that

$$\zeta(s, \mathbf{SL}_n/\mathbf{Z}) = \prod_k \zeta(s-k)^{b(n,k)}$$

and

$$\begin{aligned} \zeta(s-1, \mathbf{SL}_n/\mathbf{Z}) &= \prod_k \zeta(s-1-k)^{b(n,k)} \\ &= \prod_k \zeta(s-k)^{b(n,k-1)}. \end{aligned}$$

Hence

$$\begin{aligned} \frac{\zeta(s-1, \mathbf{SL}_n/\mathbf{Z})}{\zeta(s, \mathbf{SL}_n/\mathbf{Z})} &= \prod_k \zeta(s-k)^{b(n,k-1)-b(n,k)} \\ &= \prod_k \zeta(s-k)^{a(n,k)} \\ &= \zeta(s, \mathbf{GL}_n/\mathbf{Z}). \end{aligned}$$

The proofs for $R = \mathbf{F}_p$ and $\mathbf{F}_1$ are exactly similar. This proves (7). To see (8) we calculate for $K \geq 1$:

$$\begin{aligned} \prod_{k=1}^{K} \zeta(s+k, \mathbf{GL}_n/R) &= \prod_{k=1}^{K} \frac{\zeta(s+k-1, \mathbf{SL}_n/R)}{\zeta(s+k, \mathbf{SL}_n/R)} \\ &= \frac{\zeta(s, \mathbf{SL}_n/R)}{\zeta(s+K, \mathbf{SL}_n/R)}. \end{aligned}$$

Hence, if

$$\lim_{K\to\infty} \zeta(s+K, \mathbf{SL}_n/R) = 1$$

we obtain the result. This property is easy to see for $R = \mathbf{Z}$ and $\mathbf{F}_p$. We prove this for $R = \mathbf{F}_1$ except for the case $n = 1$ in Theorem 4 below. In the case $R = \mathbf{F}_1$ and $n = 1$, the equality of (8) is not valid. In fact since

$$\zeta(s, \mathbf{GL}_1/\mathbf{F}_1) = \frac{s}{s-1}$$

we get

$$\prod_{k=1}^{K} \zeta(s+k, \mathbf{GL}_1/\mathbf{F}_1) = \frac{s+K}{s},$$

which does not converge as $K \to \infty$. □

4. Limit to infinity.

Proof of Theorem 4. Notice that

$$\begin{aligned} \zeta(s, X/\mathbf{F}_1) &= \prod_{k=0}^{n} (s-k)^{-a_k} \\ &= s^{-\#X(\mathbf{F}_1)} \prod_{k=0}^{n} \left(1-\frac{k}{s}\right)^{-a_k}. \end{aligned}$$

Since

$$\lim_{|s|\to\infty}\prod_{k=0}^{n}\left(1-\frac{k}{s}\right)^{-a_k}=1,$$

we obtain the result. □

5. Values of zeta functions.

Proof of Theorem 6. Since

$$\zeta(-m,\mathbf{A}^n/\mathbf{F}_1)^{-1}=-(m+n),$$

it is sufficient to show the following fact:

$$p|\#K_{2p-3}(\mathbf{F}_1) \text{ for each prime } p.$$

The case $p=2$ follows from

$$K_1(\mathbf{F}_1)=\pi_1^s\cong\mathbf{Z}/2\mathbf{Z}.$$

We show that $p|\#\pi_{2p-3}^s$ for each odd prime. First we use the result of Adams [1] and Quillen [12, 13] saying that π_{2p-3}^s has a cyclic subgroup of order $\mathrm{denom}(B_{p-1}/(2(p-1)))$ (see Ravenel [14] also), where B_k is the Bernoulli number and $\mathrm{denom}(r)$ denotes the denominator of a rational number r. Next, the famous von Staudt-Clausen theorem says that

$$p|\,\mathrm{denom}(B_k)\Longleftrightarrow(p-1)|k$$

for each prime p. In particular we see that $p|\,\mathrm{denom}(B_{p-1})$. Hence $p|\,\mathrm{denom}(B_{p-1}/(2(p-1)))$. Thus $K_{2k-3}(\mathbf{F}_1)=\pi_{2p-3}^s$ has an element of order p. □

References

[1] J. F. Adams, On the groups $J(X)$. IV, Topology **5** (1966), 21–71.
[2] A. Deitmar, Schemes over $\mathbf{F}_1$, (2004). (Preprint).
[3] N. Kurokawa, Multiple zeta functions: an example, in *Zeta functions in geometry* (*Tokyo*, 1990), 219–226, Adv. Stud. Pure Math., 21, Kinokuniya, Tokyo.
[4] N. Kurokawa, Absolute Frobenius operators, Proc. Japan Acad., **80A** (2004), no. 9, 175–179.
[5] N. Kurokawa, Values of absolute tensor products, Proc. Japan Acad., **81A** (2005), no. 10, 185–190.
[6] N. Kurokawa and S. Koyama, Values of multiple zeta functions. (In preparation).
[7] N. Kurokawa, H. Ochiai and M. Wakayama, Absolute derivations and zeta functions, Doc. Math. **2003**, Extra Vol., 565–584 (electronic).
[8] S. Lichtenbaum, Values of zeta-functions, étale cohomology, and algebraic K-theory, in *Algebraic K-theory*, II: "*Classical*" *algebraic K-theory and connections with arithmetic* (*Proc. Conf.*, *Battelle Memorial Inst.*, *Seattle*, *Wash.*, 1972), 489–501. Lecture Notes in Math., 342, Springer, Berlin.
[9] Yu. I. Manin, Lectures on zeta functions and motives (according to Deninger and Kurokawa), Astérisque No. 228 (1995), 4, 121–163.
[10] Yu. I. Manin, The notion of dimension in geometry and algebra, (2005). (Preprint). `math.AG/0502016`.
[11] D. Quillen, On the cohomology and K-theory of the general linear groups over a finite field, Ann. of Math. (2) **96** (1972), 552–586.
[12] D. Quillen, The Adams conjecture, Topology **10** (1971), 67–80.
[13] D. Quillen, Letter from Quillen to Milnor on $\mathrm{Im}(\pi_i 0\to\pi_i^s\to K_i\mathbf{Z})$, in *Algebraic K-theory* (*Proc. Conf.*, *Northwestern Univ.*, *Evanston*, Ill., 1976), 182–188. Lecture Notes in Math., 551, Springer, Berlin.
[14] D. C. Ravenel, *Complex cobordism and stable homotopy groups of spheres*, Academic Press, Orlando, FL, 1986.
[15] C. Soulé, Les variétés sur le corps à un élément, Mosc. Math. J. **4** (2004), no. 1, 217–244, 312.