Certain rings whose simple singular modules are *GP*-injective

By Jin Yong Kim

Department of Mathematics and Institute of Natural Sciences, Kyung Hee University, Suwon 449-701, South Korea

(Communicated by Heisuke Hironaka, M. J. A., Sept. 12, 2005)

Abstract: We prove that if R is an idempotent reflexive left Goldie ring whose simple singular left R-modules are GP-injective, then R is a finite product of simple left Goldie rings. As a byproduct of this result we are able to show that if R is semiprime, left Goldie and left weakly π-regular, then R is a finite product of simple left Goldie rings.

Key words: Generalized principally injective module; idempotent reflexive ring; simple singular module; von Neumann regular ring; Goldie ring.

Throughout this paper, R denotes an associative ring with identity and R-modules are unital. $J(R)$ and $Z_l(R)$ denote the Jacobson radical and left singular ideal of R. A left R-module M is called *generalized left principally injective* (briefly *left GP-injective*) if, for any $0 \neq a \in R$, there exists a positive integer $n = n(a)$ such that $a^n \neq 0$ and any left R-homomorphism of Ra^n into M extends to one of ${}_RR$ into M. Note that GP-injective modules defined here are also called *YJ-injective* modules in [4, 11, 13–15]. The concept of GP-injective modules was introduced in [13] to study von Neumann regular rings, V-rings, self-injective rings and their generalizations. Actually, many authors investigated von Neumann regularity of rings whose simple left R-modules (resp. simple singular left R-modules) are GP-injective [3, 4, 6, 9, 11, 14, 15]. Mason [7] introduced the concept of reflexive ideals. As a nontrivial generalization of a reflexive ring, idempotent reflexive ring is defined here. In this paper idempotent reflexive ring whose simple singular left R-modules are GP-injective is studied. As a byproduct of this study one of the main results on weakly π-regularity of rings [5, Theorem 15] is extended. Let X be a nonempty subset of R, then $l(X)$ denotes the left annihilator of X in R.

Recall that a ring R is called *left weakly continuous* [10] if $J(R) = Z_l(R)$, $R/J(R)$ is regular and idempotents can be lifted modulo $J(R)$. Every von Neumann regular ring is left weakly continuous. It is easy to see that R is von Neumann regular if and only if R is a left weakly continuous and left PP-ring (every principal left ideal is projective). We start with the following Lemma due to Ming.

2000 Mathematics Subject Classification. 16D50, 16E50.

Lemma 1. *If $Z_l(R)$ contains no nonzero nilpotent element, then $Z_l(R) = 0$.*

Proof. See [12, Lemma 2.1]. □

Theorem 2. *For a ring R, the following statements are equivalent.*

(1) *R is von Neumann regular.*
(2) *R is left weakly continuous ring whose simple singular left R-modules are GP-injective.*

Proof. (1) ⇒ (2): Observe that if R is von Neumann regular then every left R-module is GP-injective [9, Lemma 8]. So we are done.

(2) ⇒ (1): Suppose that $Z_l(R) \neq 0$. Then by Lemma 1, we may assume that $Z_l(R)$ is not reduced. So there exists nonzero $a \in Z_l(R)$ such that $a^2 = 0$. We claim that $Z_l(R) + l(a) = R$. If not, there exists a maximal essential left ideal M containing $Z_l(R) + l(a)$. Thus R/M is GP-injective and so any left R-homomorphism from Ra to R/M extends to an R-homomorphism from R to R/M. Let $f\colon Ra \to R/M$ be defined by $f(ra) = r + M$. Then f is well-defined R-homomorphism. So there exists $r \in R$ such that $1 + M = f(a) = ar + M$. Hence $1 - ar \in M$; whence $1 \in M$, which is a contradiction. Therefore $Z_l(R) + l(a) = R$. Hence we can write $1 = c + d$ for some $c \in Z_l(R)$ and $d \in l(a)$. Thus $a = ca$ and so $(1 - c)a = 0$. Since $c \in Z_l(R) = J(R)$, $1 - c$ is invertible. Thus $a = 0$, which is also contradiction. Therefore $Z_l(R)$ is reduced and so $Z_l(R) = 0$. □

Corollary 3. *A ring R is left continuous (resp. left self-injective) regular if and only if R is left continuous (resp. left self-injective) ring whose simple singular left R-modules are GP-injective.*

A left ideal I is said to be *reflexive* [7] if $aRb \subseteq I$ implies $bRa \subseteq I$ for $a, b \in R$. A ring R is called reflexive if 0 is a reflexive ideal. We will introduce the concept of idempotent reflexive ring and give an example of a ring which is idempotent reflexive, but not reflexive.

Definition 4. A left ideal I is called *idempotent reflexive* if $aRe \subseteq I$ implies $eRa \subseteq I$ for $a, e = e^2 \in R$. We shall say R is *idempotent reflexive ring* when 0 is an idempotent reflexive ideal.

Note that any prime ideal is reflexive. Since an intersection of reflexive left ideals is reflexive, all semiprime ideals are reflexive. Recall that a ring R is said to be *abelian* if every idempotent of R is central. Obviously any abelian rings and semiprime rings are idempotent reflexive rings.

Example 5. There is an idempotent reflexive ring which is not reflexive. This example is essentially due to Birkenmeier, Kim and Park [1, Example 2.8].

Assume that $F\{X, Y\}$ is the free algebra over a field F generated by X and Y, and $\langle YX\rangle$ is the two-sided ideal of $F\{X, Y\}$ generated by the element YX. Let $R = F\{X, Y\}/\langle YX\rangle$. Put $x = X + \langle YX\rangle$ and $y = Y + \langle YX\rangle$ in R. Then $R = \{f_0(x) + f_1(x)y + \cdots + f_n(x)y^n \mid n = 0, 1, 2, \ldots,$ and $f_i(x) \in F[x]\}$, the polynomial ring such that $yx = 0$. Now let α, β be nonzero elements in R satisfying $\alpha\beta = 0$. Say $\alpha = f_0(x) + f_1(x)y + \cdots + f_n(x)y^n$ and $\beta = g_0(x) + g_1(x)y + \cdots + g_m(x)y^m$ with $f_n(x) \neq 0$ and $g_m(x) \neq 0$.

Case 1: $f_0(x) = 0$. Then $\alpha x\beta = f_0(x)x\beta = 0$. From the fact that $yg(x) = g(0)y$ for $g(x) \in F[x]$, it can be checked that $g_0(0) = g_1(0) = \cdots = g_m(0) = 0$. Thus $\alpha y\beta = \alpha(g_0(0) + g_1(0)y + \cdots + g_m(0)y^m)y = 0$. Thus $\alpha R\beta = 0$.

Case 2: $g_0(x) = 0$. Of course we may assume that $f_0(x) \neq 0$. In this case, it also can be checked that $g_1(x) = g_2(x) = \cdots = g_m(x) = 0$, a contradiction to $g_m(x) \neq 0$.

From these we have $\alpha\beta = 0$ implies $\alpha R\beta = 0$ for $\alpha, \beta \in R$. So it is easily checked that R is an abelian ring. Hence R is an idempotent reflexive ring. But R is not reflexive since $xRy \neq 0$ and $yRx = 0$.

Recall that an element $a \in R$ is called a *left weakly regular* element if $a \in RaRa$.

Lemma 6. *Let R be an idempotent reflexive ring. If $a \in R$ is not a left weakly regular element, then every maximal left ideal M of R containing $RaR + l(a)$ must be essential left ideal of R.*

Proof. Assume that $a \in R$ is not a left weakly regular element. Then $RaR + l(a)$ is a proper left ideal of R. Let M be a maximal left ideal containing $RaR + l(a)$. If M is not essential, then $M = Re$ for some $e = e^2 \in R$. Thus, $aR(1 - e) = 0$, so $(1 - e)Ra = 0$ since R is idempotent reflexive. Hence $1 - e \in l(a) \subseteq M$, so $1 \in M$. It is a contradiction. □

Using this lemma, we give here a comprehensive proof of the following proposition that slightly extends results of Xue [11, Proposition 2] and Chen and Ding [3, Lemma 4.1].

Proposition 7. *Let R be an idempotent reflexive ring. If every simple singular left R-module is GP-injective, then for any nonzero element $a \in R$, there exists a positive integer $n = n(a)$ such that $a^n \neq 0$ and $RaR + l(a^n) = R$. Consequently, $J(R) = 0$.*

Proof. If $a \in R$ is a left weakly regular element then we are done. So we may assume that a is not a left weakly regular element. Hence $RaR + l(a) \neq R$. First we assume that a is nilpotent with $a^m \neq 0$ and $a^{m+1} = 0$. Then we are able to show that $RaR + l(a^m) = R$. If not, there exists a maximal left ideal M containing $RaR + l(a^m)$. By Lemma 6, M must be an essential left ideal of R. Therefore R/M is GP-injective, and $(a^m)^2 = 0$, so any R-homomorphism of Ra^m into R/M extends to one of R into R/M. Let $f \colon Ra^m \to R/M$ be defined by $f(ra^m) = r + M$. Then f is well-defined R-homomorphism. Since R/M is GP-injective, there exists $c \in R$ such that $1 + M = f(a^m) = a^m c + M$. Since $a^m c \in M$ we obtain $1 \in M$, a contradiction. Therefore we have $RaR + l(a^m) = R$. It remains to show that the case when a is not nilpotent element of R. Consider the chain $RaR + l(a) \subseteq RaR + l(a^2) \subseteq \cdots$. Let $\bigcup_{i=1}^{\infty}[RaR + l(a^i)] = I$. If $I \neq R$, then I is contained in a maximal left ideal M of R. Again by Lemma 6, M must be an essential left ideal of R. Thus R/M is GP-injective. So there exists a positive integer n such that every R-homomorphism $Ra^n \to R/M$ extends to one of R into R/M. Define $f \colon Ra^n \to R/M$ via $ra^n \mapsto r + M$. By a similar way as in the previous process, we obtain a contradiction. Therefore we have $\bigcup_{i=1}^{\infty}[RaR + l(a^i)] = R$. Since $1 \in R$, $RaR + l(a^k) = R$ for some positive integer k. Finally, assume that $J(R) \neq 0$. Then for each nonzero $a \in J(R)$, we have $(1 - x)a^n = 0$ where $x \in RaR \subseteq J(R)$ and $a^n \neq 0$ for some positive integer n. Since $1 - x$ is invertible, we have $a^n = 0$. It is a contradiction. □

Corollary 8 ([3, Lemma 4.1]). *Let R be a semiprime ring or an abelian ring. If every simple singular left R-module is GP-injective, then for any nonzero $a \in R$, there exists a positive integer $n = n(a)$ such that $a^n \neq 0$ and $RaR + l(a^n) = R$.*

Corollary 9 ([11, Proposition 2]). *If every simple left R-module is GP-injective, then for any nonzero $a \in R$, there exists a positive integer $n = n(a)$ such that $a^n \neq 0$ and $RaR + l(a^n) = R$.*

Proof. Note that rings whose simple left R-module are *GP*-injective are always semiprimitive [14, Lemma 1]. □

Recall that an element $c \in R$ is *left regular*, if $xc = 0$ implies $x = 0$. A right and left regular element is called *regular*.

Theorem 10. *Let R be an idempotent reflexive left Goldie ring. If every simple singular left R-module is GP-injective, then R is a finite product of simple left Goldie rings.*

Proof. First note that for any nonzero element $a \in R$, $RaR + l(Ra)$ is an essential left ideal of R. Indeed, let $(RaR+l(Ra)) \cap I = 0$ for some left ideal I of R. Then for every element $b \in I$, $(RaR+l(Ra)) \cap Rb = 0$. Hence $aRb \subseteq aR \cap Rb = 0$. Since R is semiprime by Proposition 7, we have $bRa = 0$. Therefore $b \in l(Ra)$, hence $I = 0$. By [2, Theorem 1.10], $RaR + l(Ra)$ contains a regular element $c \in R$. Now we will prove that $RaR + l(Ra) = R$ for any $a \in R$. Actually we claim that $RcR = R$. Again by Proposition 7, there exists a positive integer $n = n(c)$ such that $RcR + l(c^n) = R$. Hence $(1-x)c^n = 0$ for some $x \in RcR$. Since c^n is also a regular element, $1-x = 0$. Thus $RcR = R$. Therefore $RaR+l(Ra) = R$ for any $a \in R$. This implies that R is a left weakly regular ring. Therefore R is a finite product of simple left Goldie rings by [8, Lemma 3.1]. □

Corollary 11. *Let R be a semiprime (or an abelian) left Goldie ring. If every simple singular left R-module is GP-injective, then R is a finite product of simple left Goldie rings.*

Finally we turn our attention to weakly π-regular rings. Recall that a ring R is said to be left weakly π-regular if for every $x \in R$ there exists a positive integer n, depending on x, such that $x^n \in Rx^nRx^n$.

Theorem 12. *Let R be a semiprime left Goldie ring. If R is left weakly π-regular, then R is a finite product of simple left Goldie rings.*

Proof. By the same method as in the proof of Theorem 10, for any element $a \in R$, $RaR + l(Ra)$ is an essential left ideal of R. Then $RaR + l(Ra)$ contains a regular element $c \in R$. Since R is left weakly π-regular, there exists a positive integer n such that $c^n \in Rc^nRc^n$ and so $c^n = dc^n$ for some $d \in Rc^nR$. Since c^n is also regular element and so $d = 1$. Hence $RaR + l(Ra) = R$; whence R is a left weakly regular ring. Again by [8, Lemma 3.1], R is a finite product of simple left Goldie rings. □

Corollary 13 ([5, Theorem 15]). *Let R be a prime left Goldie ring. If R is left weakly π-regular, then R is simple.*

Acknowledgments. This work was supported by the Korea Research Foundation Grant (KRF-2003-015-C00018).

References

[1] G.F. Birkenmeier, J.Y. Kim and J.K. Park, A characterization of minimal prime ideals, Glasgow Math. J. **40** (1998), no. 2, 223–236.

[2] A.W. Chatters and C.R. Hajarnavis, *Rings with chain conditions*, Pitman, Boston, Mass., 1980.

[3] J. Chen and N. Ding, On generalizations of injectivity, in *International Symposium on Ring Theory* (*Kyongju, 1999*), 85–94, Birkhäuser, Boston, Boston, MA, 2001.

[4] N.Q. Ding and J.L. Chen, Rings whose simple singular modules are YJ-injective, Math. Japon. **40** (1994), no. 1, 191–195.

[5] C.Y. Hong, N.K. Kim, T.K Kwak and Y. Lee, On weak π-regularity of rings whose prime ideals are maximal, J. Pure Appl. Algebra **146** (2000), no. 1, 35–44.

[6] N.K. Kim, S.B. Nam and J.Y. Kim, On simple singular GP-injective modules, Comm. Algebra **27** (1999), no. 5, 2087–2096.

[7] G. Mason, Reflexive ideals, Comm. Algebra **9** (1981), no. 17, 1709–1724.

[8] G.O. Michler and O.E. Villamayor, On rings whose simple modules are injective, J. Algebra **25** (1973), 185–201.

[9] S.B. Nam, N.K. Kim and J.Y. Kim, On simple GP-injective modules, Comm. Algebra **23** (1995), no. 14, 5437–5444.

[10] W.K. Nicholson and M.F. Yousif, Weakly continuous and C2-rings, Comm. Algebra **29** (2001), no. 6, 2429–2446.

[11] W. Xue, A note on YJ-injectivity, Riv. Mat. Univ. Parma (6) **1** (1998), 31–37.

[12] R. Yue Chi Ming, On von Neumann regular rings. III, Monatsh. Math. **86** (1978/79), no. 3, 251–257.

[13] R. Yue Chi Ming, On regular rings and Artinian rings. II, Riv. Mat. Univ. Parma (4) **11** (1985), 101–109.

[14] R. Yue Chi Ming, On *p*-injectivity and generalizations, Riv. Mat. Univ. Parma (5) **5** (1996), 183–188.

[15] R. Yue Chi Ming, A note on YJ-injectivity, Demonstratio Math. **30** (1997), no. 3, 551–556.