Distribution of units of algebraic number fields with only one fundamental unit

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Abstract: For some algebraic number fields F with only one fundamental unit, we give a lower bound of the extension degree of the ray class field of conductor a rational prime p over the Hilbert class field of F.

Key words: Algebraic number field; ray class field; unit; distribution.

We studied the distribution of units modulo prime ideals [K1]. We would like to generalize it. To explain our line, let us introduce notations. For an algebraic number field L, o_L, d_L denote the maximal order and the discriminant of L respectively, and let $o_L^\times$ be the group of units of L. If L is a Galois extension field of K and $\mathfrak{P}$ is a prime ideal of L, $\sigma_{L/K}(\mathfrak{P})$ denotes the Frobenius automorphism in $\mathrm{Gal}(L/K)$ with respect to $\mathfrak{P}$, and for a prime ideal $\mathfrak{p}$ of K lying below $\mathfrak{P}$ $\sigma_{L/K}(\mathfrak{p})$ denotes the conjugacy class of $\sigma_{L/K}(\mathfrak{P})$. ζ_m stands for a primitive mth root of unity. For an integral ideal $\mathfrak{n}$ of an algebraic number field F, we set

$$E(\mathfrak{n}) := \{u \bmod \mathfrak{n} \mid u \in o_F^\times\} \ (\subset (o_F/\mathfrak{n})^\times),$$
$$I(\mathfrak{n}) := [(o_F/\mathfrak{n})^\times : E(\mathfrak{n})],\ e(\mathfrak{n}) := \#E(\mathfrak{n}).$$

We note that the extension degree of the ray class field $F(\mathfrak{n})$ of conductor $\mathfrak{n}$ over F is the product of $I(\mathfrak{n})$ and the class number of F.

Regarding the study of the distribution of indices $I(\mathfrak{n})$ as that of units, we consider the following situation: Let F be an algebraic number field, and $\mathfrak{S}$ a set of integral ideals of F. To study the behavior of $I(\mathfrak{n})$ $(\mathfrak{n} \in \mathfrak{S})$ we suppose that

> natural numbers $n(\mathfrak{n})$, $f(\mathfrak{n})$ correspond to every ideal $\mathfrak{n} \in \mathfrak{S}$ and for a square-free natural number m, there is a union of conjugacy classes $H_m \subset \mathrm{Gal}(F_m/F_\mathfrak{S})$, where F_m is a Galois extension field of an algebraic number field $F_\mathfrak{S}$ dependent only on $\mathfrak{S}$.
> They satisfy that $f(\mathfrak{n})$ divides $I(\mathfrak{n})$, and that the number of ideals $\mathfrak{n} \in \mathfrak{S}$ (with possibly finitely many exceptions dependent on m) satisfying both $n(\mathfrak{n}) \le x$ and $m \mid I(\mathfrak{n})/f(\mathfrak{n})$ is equal to the number of prime ideals $\mathfrak{p}$ of $\deg \mathfrak{p} = 1$ in $F_\mathfrak{S}$ satisfying $N_{F_\mathfrak{S}/\mathbf{Q}}(\mathfrak{p}) \le x$ and $\sigma_{F_m/F_\mathfrak{S}}(\mathfrak{p}) \in H_m$ multiplied by some positive number c.

The last demand is to invoke Chebotarev's density theorem. Under this arithmetic situation, we have, for a positive number x and $X := \max_{n(\mathfrak{n})\le x} I(\mathfrak{n})/f(\mathfrak{n})$

$$\begin{aligned}\#\{&\mathfrak{n} \in \mathfrak{S} \mid n(\mathfrak{n}) \le x,\ f(\mathfrak{n}) = I(\mathfrak{n})\}\\ &= \sum_{\mathfrak{n}, n(\mathfrak{n})\le x}\ \sum_{m \mid I(\mathfrak{n})/f(\mathfrak{n})} \mu(m)\\ &= \sum_{m \le X} \mu(m)\#\{\mathfrak{n} \mid n(\mathfrak{n}) \le x,\ m \mid I(\mathfrak{n})/f(\mathfrak{n})\}\end{aligned}$$

and then Chebotarev's density theorem implies

$$\begin{aligned}\#\{&\mathfrak{n} \mid n(\mathfrak{n}) \le x,\ m \mid I(\mathfrak{n})/f(\mathfrak{n})\}\\ &= c\#H_m/[F_m : F_\mathfrak{S}] \cdot \mathrm{Li}(x) + o(\mathrm{Li}(x)).\end{aligned}$$

If the expected density

$$\kappa := \sum_m \mu(m)\#H_m/[F_m : F_\mathfrak{S}]$$

is convergent to a positive number, we may expect

$$\#\{\mathfrak{n} \mid n(\mathfrak{n}) \le x,\ f(\mathfrak{n}) = I(\mathfrak{n})\}/\mathrm{Li}(x) \to c\kappa.$$

To follow the procedure, we have, besides the construction of the situation above, an analytic difficulty, i.e. the estimate for the accumulation of error terms by occasion of making use of Chebotarev's theorem for infinitely many different fields F_m.

In [K1], we succeeded in the construction of the arithmetic situation when $\mathfrak{S}$ is some set of prime ideals. Let us explain it briefly. Let F be an objec-

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tive algebraic number field and fix any supplementary Galois extension field K of the rational number field $\mathbf{Q}$ which contains the field F, and choose and fix any element η of $\mathrm{Gal}(K/\mathbf{Q})$. Then we take $\mathfrak{S}$ as the set of prime ideals $\mathfrak{p}$ of F such that $\mathfrak{p} \nmid 2dK$ and the Frobenius automorphism of some prime ideal of K lying above $\mathfrak{p}$ is equal to η. To find the function f, we take a primitive integral polynomial $g(x)$ of minimal degree such that $\{\epsilon^{g(\eta)} \mid \epsilon \in o_F^\times\}$ is a finite group, whose order is denoted by δ_1. Then $g(x)$ divides $x^d - 1$ in $\mathbf{Z}[x]$ for $d := [\langle\eta\rangle : \langle\eta\rangle \cap \mathrm{Gal}(K/F)]$, and we put $h(x) = (x^d-1)/g(x)$. Next we take the maximal natural number δ_0 such that $\sqrt[\delta_0]{\epsilon}^{\,\delta_1 g(\rho)} = 1$ holds for $^\forall\epsilon \in o_F^\times$ and for any extension automorphism ρ of η. Then we showed that an integer $f(\mathfrak{p}) := \delta_0 h(p)/\delta_1$ divides $I(\mathfrak{p})$ for $\mathfrak{p} \in \mathfrak{S}$ where $p := n(\mathfrak{p})$ is the prime number lying below $\mathfrak{p}$, and defining $F_m, H_m, F_{\mathfrak{S}}$ somehow, we constructed the arithmetic situation and we conjectured in [K1]

$$\#\{\mathfrak{p} \in \mathfrak{S} \mid n(\mathfrak{p}) \le x, f(\mathfrak{p}) = I(\mathfrak{p})\} = \kappa\mathrm{Li}(x) + o(\mathrm{Li}(x))$$

and we showed $\kappa > 0$ by taking advantage of the existence of some automorphism. The asymptotic equation above holds under G.R.H. in case that F is a real quadratic field and in other some cases [CKY, K2, L, M, R].

As a next step, we would like to consider the case that $\mathfrak{S}$ is a subset of rational primes. In this note, we take up as F real quadratic fields, real cubic fields with $d_F < 0$ and imaginary abelian quartic fields. The rank of $o_F^\times$ is one for these fields and our main aim is to construct the arithmetic situation above and in some cases to make them result in a particular case in [K1]. Then the positivity of κ can be shown as in [K1].

For a general algebraic number field F, the case where prime numbers remain prime in F is in [K1]. Thus we omit the case here.

The details will appear elsewhere.

1. Real quadratic fields. Let F be a real quadratic field with fundamental unit ϵ. The letter p denotes odd prime numbers which split in F and let $\mathfrak{S}$ be the set of these prime numbers. We make this case end in the known case [L, M, R, K1].

1.1. The case of $N_{F/\mathbf{Q}}(\epsilon) = 1$. In this subsection, we assume $N_{F/\mathbf{Q}}(\epsilon) = 1$.

Theorem 1. $e((p)) \mid p-1$, *which is equivalent to* $p - 1 \mid I((p))$ *holds.* $I((p)) = p - 1$ *holds if and only if* $I(\mathfrak{p}) = 1$ *where* $\mathfrak{p}$ *stands for any prime ideal lying above* p.

This follows from $e(\mathfrak{p}) = p-1 \Leftrightarrow e((p)) = p-1$, making use of the order of $\epsilon \bmod \mathfrak{p}$ = the order of $\epsilon \bmod \mathfrak{p}'$. By the theorem, we see

$$\#\{p \le x \mid p \in \mathfrak{S},\ I((p)) = p-1\}$$

is equal to a half of the number of prime ideals $\mathfrak{p}$ satisfying $\deg(\mathfrak{p}) = 1$, $N_{F/\mathbf{Q}}(\mathfrak{p}) \le x$, $\mathfrak{p} \nmid 2d_F$ and $I(\mathfrak{p}) = 1$.

And then it is known that the number of prime ideals $\mathfrak{p}$ above is asymptotically equal to $\kappa\mathrm{Li}(x)$ for a positive number κ under G.R.H. by [L, M, R].

1.2. The case of $N_{F/\mathbf{Q}}(\epsilon) = -1$. In this subsection, we assume $N_{F/\mathbf{Q}}(\epsilon) = -1$. We need the following

Lemma 1. *If the order of* $\epsilon \bmod (p)$ *is* $p-1$ *and* $\epsilon^{(p-1)/2} \not\equiv \pm 1 \bmod (p)$ *holds, then* $p \equiv 3 \bmod 4$ *and the order of* $\epsilon \bmod \mathfrak{p}$ *is* $p-1$ *or* $(p-1)/2$, *where* $\mathfrak{p}$ *is any prime ideal lying above* p.

Theorem 2. $e((p)) \mid 2(p-1)$, *which is the same as* $(p-1)/2 \mid I((p))$ *holds, and* $I((p)) = (p-1)/2$ *holds if and only if both* $p \equiv 3 \bmod 4$ *and* $I(\mathfrak{p}) = 1$ *hold for any prime ideal* $\mathfrak{p}$ *lying above* p.

$e((p)) \mid 2(p-1)$ is almost obvious, and $e((p)) = 2(p-1)$ holds if and only if the order of $\epsilon \bmod (p)$ is $p-1$ and -1 is not in the group $\langle\epsilon \bmod (p)\rangle$. Then the lemma leads us to the theorem.

Set $K = F(\sqrt{-1})$ and define η in $\mathrm{Gal}(K/\mathbf{Q})$ by $\sqrt{-1}^{\,\eta} = -\sqrt{-1}$ and η = the identity on F. Then the theorem implies that

$$\#\{p \le x \mid p \in \mathfrak{S}, I((p)) = (p-1)/2\}$$

is equal to a half of the number of prime ideals $\mathfrak{p}$ of F such that $\deg(\mathfrak{p}) = 1$, $N_{F/\mathbf{Q}}(\mathfrak{p}) \le x$, $\mathfrak{p} \nmid 2d_F$, $I(\mathfrak{p}) = 1$, and $\sigma_{K/\mathbf{Q}}(\mathfrak{P})$ coincides η where $\mathfrak{P}$ is some prime ideal of K lying above $\mathfrak{p}$.

This is a special case in [K1], and it is conjectured that the number of prime ideals $\mathfrak{p}$ above is asymptotically equal to $\kappa\mathrm{Li}(x)$, and it is shown that the expected density κ is positive.

As a remark to this section, these may suggest with the case of non-decomposable primes [CKY, K1, R] that the basic part of $I((p))$ is $p-1$ or $(p-1)/2$ according as $N_{F/\mathbf{Q}}(\epsilon) = 1$ or $= -1$. The field corresponding to $(p-1)/2$ is $\mathbf{Q}(\zeta_p + \zeta_p^{-1})$. See [K3] about the quadratic extension of the composite field of the Hilbert class field and $\mathbf{Q}(\zeta_p + \zeta_p^{-1})$ in the case of $N_{F/\mathbf{Q}}(\epsilon) = 1$.

2. Real cubic fields with $d_F < 0$. In this section, F is a real cubic field with $d_F < 0$, and ϵ denotes a positive fundamental unit of F. We set $K = F(\sqrt{d_F})$, $F_m = K(\sqrt[m]{o_K^\times})$.

2.1. The case of $(p) = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3$. In this subsection, let $\mathfrak{S}$ be the set of odd primes p which split fully in F, and the letter p denotes an element in $\mathfrak{S}$.

Lemma 2. *$\epsilon^{(p-1)/2} \not\equiv -1 \bmod (p)$ holds. For a divisor m of $p-1$, the order r_p of $\epsilon \bmod (p)$ divides $(p-1)/m$ if and only if $\zeta_m^{\rho-1} = \sqrt[m]{\epsilon}^{\rho-1} = 1$ holds for $\rho = \sigma_{F_m/\mathbf{Q}}(\mathfrak{P})$, where $\mathfrak{P}$ is any prime ideal of F_m lying above p.*

This can be proven by translating the condition $\epsilon^{(p-1)/m} \equiv 1 \bmod (p)$ in terms of Frobenius automorphisms. Using the lemma, we can define the set H_m by the subset of automorphisms $\rho \in \mathrm{Gal}(F_m/\mathbf{Q})$ which is the identity on $K(\zeta_m, \sqrt[m]{\epsilon_1}, \sqrt[m]{\epsilon_2}, \sqrt[m]{\epsilon_3})$, where ϵ_i are conjugates of ϵ, and the key theorem is

Theorem 3. *$e((p)) \mid 2(p-1)$, which is equivalent to $(p-1)^2/2 \mid I((p))$ holds, and then $I((p)) = (p-1)^2/2$ holds if and only if $r_p = p-1$ holds.*

In this case, we must modify the argument in the situation as follows:

The number of

$$
\begin{aligned}
&\{p \in \mathfrak{S} \mid p \le x,\, I((p)) = (p-1)^2/2\} \\
&= \#\{p \in \mathfrak{S} \mid p \le x, r_p = p-1\} \\
&= \sum_{p\in\mathfrak{S}, p\le x} \sum_{m\mid (p-1)/r_p} \mu(m) \\
&= \sum_{m\le x} \mu(m)\#\{p \in \mathfrak{S} \mid p \le x, r_p \mid (p-1)/m\} \\
&= \sum_{m\le x} \mu(m)\#\{p \le x \mid \sigma_{F_m/\mathbf{Q}}((p)) \in H_m\}.
\end{aligned}
$$

And we can show that

$$\kappa = \sum_m \mu(m)\#H_m/[F_m : \mathbf{Q}] > 0$$

as in [K1].

Hence we conjecture that the number of $\{p \in \mathfrak{S} \mid p \le x,\, I((p)) = (p-1)^2/2\}$ is asymptotically equal to $\kappa\mathrm{Li}(x)$.

2.2. The case of $(p) = \mathfrak{p}_1\mathfrak{p}_2$. In this subsection, we assume that p denotes odd prime numbers which split as $(p) = \mathfrak{p}_1\mathfrak{p}_2$ in F of $\deg \mathfrak{p}_i = i$, i.e. p remains prime in $\mathbf{Q}(\sqrt{d_F})$, and let $\mathfrak{S}$ be the set of such primes. Let $\eta \in \mathrm{Gal}(K/\mathbf{Q})$ be an automorphism of order 2 satisfying $\epsilon^{\eta-1} \ne 1$. If F is pure cubic and $K(\sqrt[3]{\epsilon})$ is a Galois extension of $\mathbf{Q}$, then we set $\delta_0 = 3$, otherwise $\delta_0 = 1$. $\delta_0 \mid I(\mathfrak{p}_2)$ is known [K2, K4].

Theorem 4. *If $I(\mathfrak{p}_2)$ is odd, then $I((p)) = (p-1)/2 \cdot I(\mathfrak{p}_2)$.*

Making use of the fact that the order of $\epsilon \bmod \mathfrak{p}_1$ divides the order of $\epsilon \bmod \mathfrak{p}_2$ ([K2]), we can reduce the argument on p to that on $\mathfrak{p}_2$, and show that $e(\mathfrak{p}_2) = (p^2-1)/\delta \Leftrightarrow e((p)) = 2(p^2-1)/\delta$ for an odd natural number δ, which yields the theorem.

By this,

$$\#\{p \in \mathfrak{S} \mid p \le x,\, I((p)) = (p-1)/2 \cdot \delta_0\}$$

is equal to the number of primes $p\,(\le x)$ such that $I(\mathfrak{p}_2) = \delta_0$ and $\sigma_{K/\mathbf{Q}}(\mathfrak{P}) = \eta$ for a prime ideal $\mathfrak{P}$ of K lying above $\mathfrak{p}_2$. (The last condition on $\mathfrak{P}$ is automatically satisfied.)

This is a case treated in [K1, K2, K4], and there κ is already shown to be positive.

As a remark to this section, with [K1, K2], we can say that the basic part of $I((p))$ is $(p-1)^2/2$ if p splits fully in F, otherwise $\delta_0(p-1)/2$. In the former case, $p-1$ divides the extension degree of the ray class field of conductor (p) over the composite field of the Hilbert class field of F and $\mathbf{Q}(\zeta_p + \zeta_p^{-1})$.

3. Imaginary quartic abelian fields. In this section, F is an imaginary quartic abelian field, and F_0 is the unique real quadratic subfield. ϵ (resp. $\epsilon_0\,(>0)$) is a fundamental unit of F (resp. F_0). W denotes the group of roots of unity in F, and set $w = \#W$ and $Q = [o_F^\times : W o_{F_0}^\times]$. We set $F_m = F(\sqrt[m]{o_F^\times})$ for a natural number m.

Lemma 3. *$Q = 1$ or 2 holds, and if $Q = 1$ then we may assume $\epsilon = \epsilon_0$. Otherwise, we may assume $\epsilon_0 = \zeta_w\epsilon^2$, $\overline{\epsilon} = \zeta_w\epsilon$ and $N_{F_0/\mathbf{Q}}(\epsilon_0) = 1$.*

This is well-known.

Lemma 4. *Let p be a prime number such that $p \nmid 2d_F$, and r_p the minimal natural number such that $\epsilon^{r_p} \equiv \zeta_w^a \bmod (p)$ for some integer a. Then $e((p)) = wr_p$ holds, and if $\epsilon^r \equiv \zeta_w^a \bmod (p)$ for some natural number a, then r_p divides r.*

This is obvious.

3.1. The case of $(p) = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3\mathfrak{p}_4$. In this subsection, $\mathfrak{S}$ consists of odd prime numbers which split fully in F. We set $\Delta = 2$ if both $Q = 1$ and either $N_{F_0/\mathbf{Q}}(\epsilon_0) = 1$ or $\sqrt{-1} \in F$. Otherwise, we set $\Delta = 1$.

Proposition 1. $\Delta = 2$ *holds if and only if* $\epsilon^{(p-1)/2} \equiv \pm 1 \bmod (p)$ *holds for every prime number* p *in* $\mathfrak{S}$.

To show this, we need $\epsilon^{(p-1)/2} \equiv \delta_p (= \pm 1) \bmod (p) \Leftrightarrow \sqrt{\epsilon}^{\rho-1} = \delta_p$ for the Frobenius automorphism ρ of every prime ideal $\mathfrak{P}$ of F_2 lying above $p \Leftrightarrow \zeta_{2w}^{\rho} = \zeta_{2w}$ holds if $Q = 2$, and either $N_{F_0/\mathbf{Q}}(\epsilon_0) = 1$ or $\sqrt{-1}^{\rho} = \sqrt{-1}$ holds if $Q = 1$.

The proposition implies $e((p)) \mid w(p-1)/\Delta$, which is equivalent to $f((p)) := (p-1)^3\Delta/w \mid I((p))$. Analyzing the condition $m \mid I((p))/f((p))$ in terms of Frobenius automorphisms, we reach the following definition of H_m: For a square-free natural number m, we denote by H_m the set of automorphisms ρ in $\mathrm{Gal}(F_{2m}/F(\zeta_{2m}))$ satisfying that (i) in case of $Q = 1$, (i.1) $\sqrt[\Delta m]{\epsilon}^{\,\rho-1} = \pm 1$ holds and (i.2) $\zeta_8^{\rho-1} = 1$ holds if $\sqrt{-1} \in F$, $N_{F_0/\mathbf{Q}}(\epsilon_0) = -1$ and $2 \mid m$ hold, (ii) in case of $Q = 2$, $(\zeta_{mw} \sqrt[m]{\epsilon}^{\,2})^{\rho-1} = 1$ holds.

Theorem 5. *For* $p \in \mathfrak{S}$, $f((p)) \mid I((p))$ *holds. A square-free natural number* m *divides* $I((p))/f((p))$ *if and only if* $\sigma_{F_{2m}/\mathbf{Q}}(p)$ *is in* H_m.

Setting $n((p)) = p$ and $F_{\mathfrak{S}} = \mathbf{Q}$, we complete the construction of the arithmetic situation, and the positivity of κ is proved similarly to [K1].

3.2. The case of $(p) = \mathfrak{p}_1\mathfrak{p}_2$. In this subsection, $\mathfrak{S}$ is the set of odd prime numbers p which split as $(p) = \mathfrak{p}_1\mathfrak{p}_2$ in F with $\deg \mathfrak{p}_i = 2$. We set $\mu = 2$ if $N_{F_0/\mathbf{Q}}(\epsilon_0) = 1$ and $Q = 1$. Otherwise, we set $\mu = 1$. Moreover, we set $s_p = 1, -1$ according as p decomposes or remains prime in F_0. Set $f((p)) = (p^2-1)(p+s_p)\mu/w$.

Theorem 6. *For* $p \in \mathfrak{S}$, $e((p)) \mid (p - s_p)w/\mu$, *which is equivalent to* $f((p)) \mid I((p))$ *holds.*

Once we can guess what happens, the proof is not difficult.

Translating the condition $m \mid I((p))/f((p))$ in terms of Frobenius automorphisms, we get to the following definition of H_m: For a square-free natural number m, we define the subset H_m by the set of automorphisms $\rho \in \mathrm{Gal}(F_{m\mu}/\mathbf{Q})$ satisfying the following properties:

- the order of ρ on F is two, and $\zeta_{m\mu}^{\rho - s_\rho} = 1$ holds,
- in case of $Q = 1$, $\sqrt[m\mu]{\epsilon}^{\rho - s_\rho} = \pm 1$ holds and if moreover $N_{F_0/\mathbf{Q}}(\epsilon_0) = -1$, then $\sqrt[m]{-1}^{\rho - s_\rho} = 1$,
- in case of $Q = 2$, $(\zeta_{mw} \sqrt[m]{\epsilon}^{\,2})^{\rho - s_\rho} = 1$ and $\sqrt[m]{\epsilon}^{\,w(\rho - s_\rho)} = 1$ hold

where $s_\rho = 1$ if ρ is the identity of F_0, and $s_\rho = -1$, otherwise.

Theorem 7. *For a square-free integer* m *and* $p \in \mathfrak{S}$, m *divides* $I((p))/f((p))$ *if and only if* $\sigma_{F_{m\mu}/\mathbf{Q}}(p) \in H_m$ *holds.*

Setting $F_{\mathfrak{S}} = \mathbf{Q}$ and $n((p)) = p$, we complete the construction of the arithmetic situation.

We can show the positivity of κ similarly to [K1].

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