On the Stiefel-Whitney class of the adjoint representation of E_8

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Abstract: Let $\widetilde{E}_8$ be the 3-connected covering space of the 1-connected, compact exceptional group E_8, which is regarded as the loop space of the homotopy fibre $B\widetilde{E}_8$ of a map from BE_8, the classifying space of E_8, to an Eilenberg-MacLane space. The Stiefel-Whitney classes of the adjoint representation of E_8 induce elements of the mod 2 cohomology of $B\widetilde{E}_8$. These images are computed.

Key words: Stiefel-Whitney class; classifying space; exceptional Lie group; adjoint representation.

1. Introduction. Let E_l be the 1-connected compact exceptional Lie group of type E_l, where l is the rank. Let $\widetilde{E}_l$ be the 3-connected covering space of E_l. The cohomology modulo 2 of the classifying space of E_8 is not determined, but that of $\widetilde{E}_8$, $H^*(B\widetilde{E}_8)$, is done. Refer to [3] for it and also [2] for the action of A^*, the mod 2 Steenrod algebra. We need data to compute $H^*(BE_8)$ with spectral sequences. Our main result is Theorem 4, which states the image of the Stiefel-Whitney class of the adjoint representation of E_8 in $H^*(B\widetilde{E}_8)$. Detailed proofs will be found in another paper.

Throughout this paper $H^*(X)$ denotes the mod 2 cohomology ring of a space X. If S is a non-empty subset of an algebra, $\langle S\rangle$ denotes the subalgebra generated by S.

2. Cohomology of the classifying space of 3-connected cover. Let T^l be a maximal torus of E_l and q' a generator of $H^4(BE_l;\boldsymbol{Z})$. Let π_l, $\widehat{\pi}_l$, λ_l, $\widetilde{\lambda}_l$, φ_l and $\widetilde{\varphi}_l$ denote the natural maps such that the following diagrams are commutative, where the rows are fibrations in the left one.

$$\begin{array}{ccccc} B\widetilde{T}^l & \xrightarrow{\widehat{\pi}_l} & BT^l & \longrightarrow & K(\boldsymbol{Z},4) \\ \downarrow{\scriptstyle \widetilde{\lambda}_l} & & \downarrow{\scriptstyle \lambda_l} & & \| \\ B\widetilde{E}_l & \xrightarrow{\pi_l} & BE_l & \xrightarrow{q'} & K(\boldsymbol{Z},4) \end{array} \qquad \begin{array}{ccc} B\widetilde{E}_{l-1} & \xrightarrow{\pi_{l-1}} & BE_{l-1} \\ \downarrow{\scriptstyle \widetilde{\varphi}_l} & & \downarrow{\scriptstyle \varphi_l} \\ B\widetilde{E}_l & \xrightarrow{\pi_l} & BE_l \end{array}$$

Since $H^4(BE_l;\boldsymbol{Z}) \cong \boldsymbol{Z}$ and BE_l is 3-connected, there is a unique non-zero element q in $H^4(BE_l)$. Let $H^*(BT^l) \cong \boldsymbol{F}_2[t_1, t_2, \ldots, t_l]$. Let c_i be the i-th elementary symmetric polynomial in t_i's, and also its image in $H^*(B\widetilde{T}^l)$. Note that q is the mod 2 reduction of q' and $\lambda_l^*(q) = c_2$. Define elements c_5', c_7' and c_9' by $c_5 + c_4c_1$, $c_7 + c_6c_1$ and $c_8c_1 + c_7{c_1}^2 + c_6{c_1}^3$, respectively.

The following facts are known ([2]).

(i) $H^*(B\widetilde{T}^l) = \boldsymbol{F}_2[t_1, t_2, \ldots, t_l, \gamma_3, \gamma_5, \gamma_9, \gamma_{17}, v_{2^j+1}\ (j \geq 5)]/(c_2, c_3, c_5', c_9')$,
where $\deg \gamma_i = 2i$, $\deg v_i = i$, and $\widehat{\pi}_l^*(t_i)$ is written simply as t_i for short.

(ii) $H^*(B\widetilde{E}_6) = \boldsymbol{F}_2[y_{10}, y_{12}, y_{16}, y_{18}, y_{24}, y_{33}, y_{34}, y_{2^i+1}\ (i \geq 6)]$,
$H^*(B\widetilde{E}_7) = \boldsymbol{F}_2[y_{12}, y_{16}, y_{20}, y_{24}, y_{28}, y_{33}, y_{34}, y_{36}, y_{2^i+1}\ (i \geq 6)]$,
$H^*(B\widetilde{E}_8) = \boldsymbol{F}_2[y_{16}, y_{24}, y_{28}, y_{30}, y_{31}, y_{33}, y_{34}, y_{36}, y_{40}, y_{48}, y_{2^i+1}\ (i \geq 6)]$,
where $\deg y_i = i$.

(iii) If both $H^*(B\widetilde{E}_l)$ and $H^*(B\widetilde{E}_{l-1})$ have a generator y_i, $\widetilde{\varphi}_l^*(y_i) = y_i$. $\widetilde{\varphi}_l^*(y_i) = 0$ only when $i = 30, 31$ for $l = 8$ or $i = 28$ for $l = 7$. All the precise values of $\widetilde{\varphi}_l^*(y_i)$ are known (*cf.* (v)), and it is immediate to see that we obtain a regular sequence $(\,\widetilde{\varphi}_l^*(y_i)\,)_i$ if we exclude $\widetilde{\varphi}_l^*(y_i)$ which is null. Thus $\operatorname{Ker}\widetilde{\varphi}_7^* = (y_{28})$ and $\operatorname{Ker}\widetilde{\varphi}_8^* = (y_{30}, y_{31})$.

(iv) $\widetilde{\lambda}_l^*(y_i)$ is non-zero and contained in $\langle t_1, \ldots, t_l\rangle$, only if $i = 16, 24, 28, 30$ when $l = 8$, only if $i = 16, 24, 28$ when $l = 7$, and only if $i = 16, 24$ when $l = 6$. $\widetilde{\lambda}_l^*(y_i) = v_i$ if $i = 2^j + 1$ and $j \geq 5$, and $\widetilde{\lambda}_8^*(y_{31}) = 0$. $\widetilde{\lambda}_l^*(y_i) \in \langle t_1, \ldots, t_l, \gamma_3, \gamma_5, \gamma_9, \gamma_{17}\rangle$ in other cases. $\widetilde{\lambda}_l^*(y_i)$ is also known completely and hence $\operatorname{Ker}\widetilde{\lambda}_6^* = 0$, $\operatorname{Ker}\widetilde{\lambda}_7^* = 0$, and $\operatorname{Ker}\widetilde{\lambda}_8^* = (y_{31})$.

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Table I.

	Sq^1	Sq^2	Sq^4	Sq^8	Sq^{16}	Sq^{32}	Sq^{2^i}
y_{16}	0	0	0	y_{24}	${y_{16}}^2$	0	
y_{24}	0	0	y_{28}	0	$y_{24}y_{16}$	0	
y_{28}	0	y_{30}	0	0	$y_{28}y_{16}$	0	
y_{30}	y_{31}	0	0	0	$y_{30}y_{16}$	0	
y_{31}	0	0	0	0	$y_{31}y_{16}$	0	
y_{33}	y_{34}	0	0	0	$y_{33}y_{16}$	y_{65}	
y_{34}	0	y_{36}	0	0	$y_{34}y_{16}$	$y_{36}y_{30} + {y_{33}}^2$	
y_{36}	0	0	y_{40}	0	$y_{36}y_{16}$	$y_{40}y_{28} + {y_{34}}^2$	
y_{40}	0	0	0	y_{48}	$y_{40}y_{16}$	$y_{48}y_{24} + {y_{36}}^2$	
y_{48}	0	0	0	0	$y_{40}y_{24} + y_{36}y_{28} + y_{34}y_{30} + y_{33}y_{31}$	$y_{48}{y_{16}}^2 + {y_{40}}^2 + y_{40}y_{24}y_{16} + y_{36}y_{28}y_{16} + y_{34}y_{30}y_{16} + y_{33}y_{31}y_{16}$	
y_{12}	0	0	y_{16}	y_{20}	0	0	
y_{20}	0	0	${y_{12}}^2$	y_{28}	$y_{36} + y_{24}y_{12} + y_{20}y_{16}$	0	
y_{10}	0	y_{12}	0	y_{18}	0	0	
y_{18}	0	${y_{10}}^2$	0	0	$y_{34} + y_{24}y_{10} + y_{18}y_{16}$	0	
y_{2^i+1}	0	0	0	0	0	0 $(i \geq 6)$	$y_{2^{i+1}+1}$

(v) The action of A^* on $H^*(B\widetilde{E}_l)$ satisfies Table I. and the fact $Sq^{2^j} y_{2^i+1} = 0$ $(j < i)$.

In Table I $y_{30} = 0$, $y_{40} = y_{28}y_{12} + y_{24}y_{16} + {y_{20}}^2 + y_{16}{y_{12}}^2$ for $l = 7$, and $y_{20} = {y_{10}}^2$, $y_{28} = 0$, $y_{36} = y_{24}y_{12} + {y_{18}}^2 + y_{16}{y_{10}}^2$ for $l = 6$. (The action and $\widetilde{\varphi}_l^*(y_i)$ are determined completely.)

Lemma 1. (i) $\operatorname{Ker} \widetilde{\varphi}_7^* = (y_{28})$ *and* $\operatorname{Ker} \widetilde{\varphi}_8^* = (y_{30}, y_{31})$.
(ii) $\operatorname{Ker} \widetilde{\lambda}_6^* = 0$, $\operatorname{Ker} \widetilde{\lambda}_7^* = 0$, *and* $\operatorname{Ker} \widetilde{\lambda}_8^* = (y_{31})$.
(iii) $\operatorname{Im} \pi_6^* \subset \boldsymbol{F}_2[y_{16}, y_{24}]$, $\operatorname{Im} \pi_7^* \subset \boldsymbol{F}_2[y_{16}, y_{24}, y_{28}]$, *and* $\operatorname{Im} \pi_8^* \subset \boldsymbol{F}_2[y_{16}, y_{24}, y_{28}, y_{30}] \oplus (y_{31})$.

We show here a sketch of a proof of the last inclusion. First note that $\widetilde{\lambda}_8^*(\operatorname{Im} \pi_8^*) \subset \operatorname{Im} \widehat{\pi}_8^* \cap \operatorname{Im} \widetilde{\lambda}_8^* = \langle t_1, \ldots, t_8 \rangle \cap \operatorname{Im} \widetilde{\lambda}_8^*$. Thus $\operatorname{Im} \pi_8^*$ is contained in $\langle y_{16}, y_{24}, y_{28}, y_{30} \rangle \oplus \operatorname{Ker} \widetilde{\lambda}_8^*$. Other inclusions are proved similarly.

3. Stiefel-Whitney class. Let Ad_{E_l} be the adjoint representation of E_l. It is known that the restriction of Ad_{E_8} to E_7 satisfies $Ad_{E_8}|_{E_7} = Ad_{E_7} \oplus \lambda \oplus$ (3-dimensional trivial representation), where $\lambda : E_7 \to U(56) \to O(112)$ is a representation. (Refer to Case 2 in page 52 of [1], for example.) From Corollary 4.6, Proposition 6.1 and Corollary 6.9 of [6], and from Proposition 2.11, Theorem 2.12 and Corollary 3.7 of [5] we deduce $H^*(BE_7)$ is generated by x_4 and the Stiefel-Whitney class $w_{64}(Ad_{E_7})$ as an A^*-algebra, and also by x_4 and $w_{64}(\lambda)$, where x_4 is the generator of degree 4. The A^*-subalgebra of $H^*(BE_7)$ generated by x_4 has the trivial image in $H^*(B\widetilde{E}_7)$ via π_7^*, and also in $H^*(B\widetilde{T}^7)$. Note that by Wu formulae $\pi_7^*(w_i(Ad_{E_7})) = \pi_7^*(w_i(\lambda)) = 0$, if $i \leq 63$ or $65 \leq i \leq 95$. A similar fact holds for $H^*(BE_6)$: $H^*(BE_6)$ is generated by x_4 and $w_{32}(\mu)$ as an A^*-algebra, where μ is a representation of E_6 of degree 54. See Theorem 6.21 and Remark following it of [4].

Proposition 2. $\pi_6^*(w_{32}(\mu)) = {y_{16}}^2$, *and* $\pi_7^*(w_{64}(Ad_{E_7})) = \pi_7^*(w_{64}(\lambda)) = {y_{16}}^4$.

We sketch a proof. Lemma 1 implies that $\pi_6^*(w_{32}(\mu)) = \alpha {y_{16}}^2$, where α is a scalar. Since $H^*(BT^6)$ is a finite $H^*(BE_6)$-module, $\widehat{\pi}_6^*(H^*(BT^6))$ is also finite. If $\alpha = 0$, the image $\pi_6^*(H^*(BE_6))$ is trivial, and so in $H^*(B\widetilde{T}^6)$. This is a contradiction, and therefore $\pi_6^*(w_{32}(\mu)) = {y_{16}}^2$.

In the case of $H^*(B\widetilde{E}_7)$, $\pi_7^*(w_{64}(\lambda))$ is expressed in the form $\alpha {y_{16}}^4 + \beta {y_{24}}^2 y_{16}$ $(\alpha, \beta \in \boldsymbol{F}_2)$ by Lemma 1. Applying Sq^8, we conclude $\beta = 0$ since $\pi_7^*(w_i(\lambda)) = 0$ when $65 \leq i \leq 95$. If $\alpha = 0$, we can show a contradiction like the case of $H^*(B\widetilde{E}_6)$. By arguing similarly for $\pi_7^*(w_{64}(Ad_{E_7}))$ in addition, we obtain the result, and hence Proposition 3 below by Wu formulae.

Proposition 3. $\operatorname{Im} \pi_6^* = \boldsymbol{F}_2[{y_{16}}^2, {y_{24}}^2]$ *and* $\operatorname{Im} \pi_7^* = \boldsymbol{F}_2[{y_{16}}^4, {y_{24}}^4, {y_{28}}^4]$. *Therefore* $\operatorname{Im} \pi_8^* \subset$

$\boldsymbol{F}_2[{y_{16}}^4, {y_{24}}^4, {y_{28}}^4] \oplus y_{30} \cdot \boldsymbol{F}_2[y_{16}, y_{24}, y_{28}] \oplus (y_{31})$.

Thus $\widetilde{\varphi}_8{}^*\big(\pi_8{}^*(w_{2^i}(Ad_{E_8}))\big) = 0$ by the decomposition of $Ad_{E_8}|_{E_7}$, if $i \leq 6$. By Lemma 1 $\pi_8{}^*(w_{2^i}(Ad_{E_8})) = 0$ if $i \leq 5$, and $\pi_8{}^*(w_{64}(Ad_{E_8})) = \alpha y_{31} y_{33}$, where α is a scalar. Applying Sq^1 we conclude $\alpha = 0$. Now $\pi_8{}^*(w_{128}(Ad_{E_8}))$ is computed in a manner similar to the proof of Proposition 2. For this computation, note that $\lambda_7{}^*(w_{128}(Ad_{E_7})) = 0$, which is obtained by decomposition $Ad_{E_7}|_{T^7} = \nu \oplus$ (7-dimensional trivial representation) because of the root space decomposition, where ν is a representation of T^7 of dimension 126. This ensures that $\pi_7{}^*(w_{128}(Ad_{E_7})) = 0$ by Lemma 1.

Theorem 4. *$\pi_8{}^*(w_{2^i}(Ad_{E_8})) = 0$ for $i \leq 6$, and $\pi_8{}^*(w_{128}(Ad_{E_8})) = {y_{16}}^8$.*

Corollary 5. *$\boldsymbol{F}_2[{y_{16}}^8, {y_{24}}^8, {y_{28}}^8, {y_{30}}^8, {y_{31}}^8] \subset \operatorname{Im} \pi_8{}^* \subset \boldsymbol{F}_2[{y_{16}}^8, {y_{24}}^8, {y_{28}}^8, {y_{30}}^8, {y_{31}}^8] + Q$, where $Q \subset y_{30} \cdot \boldsymbol{F}_2[y_{16}, y_{24}, y_{28}] \oplus (y_{31})$.*

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