On class number formula for the real quadratic fields

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Abstract: Let $k > 1$ be the fundamental discriminant, and let $\chi(n)$, ε and h be the real primitive character modulo k, the fundamental unit and the class number of the real quadratic field $\mathbf{Q}(\sqrt{k})$, respectively. And let $[x]$ denote the greatest integer not greater than x.

In [3], M.-G. Leu showed $h = \left[\sqrt{k}/(2\log\varepsilon)\sum_{n=1}^{k}\chi(n)/n\right] + 1$ for all k, and $h = \left[\sqrt{k}/(2\log\varepsilon)\sum_{n=1}^{[k/2]}\chi(n)/n\right]$ in the case $k \neq m^2+4$ with $m \in \mathbf{Z}$.

In this paper we will show $h = \left[\sqrt{k}/(2\log\varepsilon)\sum_{n=1}^{[k/2]}\chi(n)/n\right]$ for all fundamental discriminants $k > 1$.

Key words: Class number; real quadratic fields.

1. Introduction. In this paper let $k > 1$ be the fundamental discriminant, and let $\chi(n)$, ε and h be the real primitive character modulo k, the fundamental unit and the class number of the real quadratic field $\mathbf{Q}(\sqrt{k})$ respectively, and $[x]$ denotes the greatest integer not greater than x.

The purpose of this paper is to show the following theorem:

Theorem 1. *We have*

$$h = \left[\frac{\sqrt{k}}{2\log\varepsilon}\sum_{n=1}^{[\frac{k}{2}]}\frac{\chi(n)}{n}\right].$$

We start the proof of Theorem 1 from the inequality in [3]

$$\sum_{n=1}^{k}\frac{\chi(n)}{n} < L(1,\chi) < \sum_{n=1}^{[k/2]}\frac{\chi(n)}{n}.$$

If the inequality

$$\left|\frac{\sqrt{k}}{2\log\varepsilon}\sum_{n=[k/2]+1}^{k}\chi(n)/n\right| < 1$$

holds, then the theorem is proved. Therefore we shall show this inequality.

Now we need the following lemmas:

Lemma 2 (Abel's identity)**.** *For any arithmetical function $a(n)$, let*

$$A(x) = \sum_{n\le x} a(n),$$

where $A(x) = 0$ if $x < 1$. Assume f has a continuous derivative on the interval $[y, x]$, where $0 < y < x$. Then we have

$$\sum_{y<n\le x} a(n)f(n) = A(x)f(x) - A(y)f(y) - \int_y^x A(t)f'(t)dt.$$

Lemma 3 (Pólya's inequality)**.**

$$\left|\sum_{n\le x}\chi(n)\right| \le \sqrt{k}\log k$$

for any primitive character $\chi(n)$ modulo k and all real number $x > 1$.

Proofs of Lemmas 2 and 3 are found in [1].

If $k \neq m^2+4$ for any integer m, then the following fact is known.

Proposition 4. *Let $k > 1$ be the fundamental discriminant, $\chi(n)$ be the real primitive character modulo k, ε and h be the fundamental unit and the class number of the real quadratic field $\mathbf{Q}(\sqrt{k})$. And let $[x]$ denote the greatest integer not greater than x. Then we have followings:*

(1) $$h = \left[\frac{\sqrt{k}}{2\log\varepsilon}\sum_{n=1}^{k}\frac{\chi(n)}{n}\right] + 1.$$

(2) *For the fundamental discriminant k such that $\sqrt{k-4} \notin \mathbf{Z}$,*

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$$h = \left[\frac{\sqrt{k}}{2 \log \varepsilon} \sum_{n=1}^{[k/2]} \frac{\chi(n)}{n} \right].$$

Proof of this proposition is seen in [3, Corollary 2 of Theorem 4].

2. Proof of Theorem 1. By Proposition 4, we shall prove the theorem for fundamental discriminant $k = m^2 + 4$, where m is an integer. In this case, since $\varepsilon = (m + \sqrt{k})/2 < \sqrt{k}$, we must estimate $\sum_{n=[k/2]+1}^{k} \chi(n)/n$ and ε a little tighter.

Let $A(t) = \sum_{n=1}^{[t]} \chi(n)$ and using Lemmas 2 and 3,

$$\begin{aligned} \left| \sum_{n=[k/2]+1}^{k} \frac{\chi(n)}{n} \right| &= \left| \int_{k/2}^{k-1} \frac{A(t)}{t^2} dt \right| \\ &\leq \sqrt{k} \log k \int_{k/2}^{k-1} \frac{1}{t^2} dt \\ &= \frac{(k-2) \log k}{\sqrt{k}(k-1)}. \end{aligned}$$

Therefore

$$\left| \frac{\sqrt{k}}{2 \log \varepsilon} \sum_{n=[\frac{k}{2}]+1}^{k} \frac{\chi(n)}{n} \right| \leq \frac{k-2}{k-1} \cdot \frac{\log k}{2 \log \varepsilon}.$$

And

$$\begin{aligned} \varepsilon^2 &= \frac{(k-2) + \sqrt{k(k-4)}}{2} \\ &> \frac{(k-2) + (k-4)}{2} \\ &= k - 3. \end{aligned}$$

Hence

$$\frac{k-2}{k-1} \cdot \frac{\log k}{2 \log \varepsilon} < \frac{k-2}{k-1} \cdot \frac{\log k}{\log (k-3)}.$$

Now we define a function $f(x)$ by

$$f(x) = (x-1) \log (x-3) - (x-2) \log x$$

for $x > 3$. Then we have

$$\begin{aligned} f'(x) &= \log(x-3) - \log x + \frac{2}{x-3} + \frac{2}{x}, \\ f''(x) &= \frac{1}{x-3} - \frac{1}{x} - \frac{2}{(x-3)^2} - \frac{2}{x^2} \\ &= \frac{-x^2 + 3x - 18}{x^2 (x-3)^2} < 0, \end{aligned}$$

and $f'(4) = 5/2 - 2 \log 2 > 0$. $f'(x) \searrow 0$ implies $f'(x) > 0$, that is, $f(x)$ is monotone increasing for $x \geq 4$.

Next,

$$\begin{aligned} f(24) &= 23 \log 21 - 22 \log 24 \\ &= \log \frac{3 \cdot 7^{23}}{8^{22}} \\ &= \log \frac{82106242020242749029}{73786976294838206464} \\ &> 0. \end{aligned}$$

Therefore $(k-2)/(k-1) \cdot (\log k)/(\log (k-3)) < 1$ for $k \geq 24$.

One can easily verify the statement for $k = 5$, 8, and 13. □

By this theorem, we see $i = 0$ for all discriminant $k = m^2 + 4$ in [3, Corollary 3 of Theorem 4].

Lemma 5. *If the discriminant of a field contains only one prime factor, then the class number of the field is odd.*

Proof is found in [2].

By Theorem 1 and Lemma 5, the class number h of the real quadratic field $\mathbf{Q}(\sqrt{p})$ is odd for a prime $p \equiv 1 \pmod 4$. Therefore we have the following corollary again ([3, Corollary 1 of Theorem 4]).

Corollary 6. *For a prime* $p \equiv 1 \pmod 4$, *the followings are equivalent*:

(1) *The class number of the real quadratic field* $\mathbf{Q}(\sqrt{p})$ *is one*;

(2) $$\frac{\sqrt{p}}{2 \log \varepsilon} \sum_{n=1}^{p} \frac{\chi(n)}{n} < 1;$$

(3) $$\frac{\sqrt{p}}{2 \log \varepsilon} \sum_{n=1}^{(p-1)/2} \frac{\chi(n)}{n} < 3.$$

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