An optimal inequality and an extremal class of graph hypersurfaces in affine geometry

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(Communicated by Shigefumi Mori, M. J. A., Sept. 13, 2004)

Abstract: We discover a general optimal inequality for graph hypersurfaces in affine $(n+1)$-space $\mathbf{R}^{n+1}$ involving the Tchebychev vector field. We also completely classify the hypersurfaces which verify the equality case of the inequality.

Key words: Optimal inequality; graph hypersurface; extremal class.

1. Introduction. A hypersurface $f: M \to \mathbf{R}^{n+1}$, $n \geq 2$, in an affine $(n+1)$-space is called a *graph hypersurface* if its affine normal vector field is some constant transversal vector field ξ. A result of Nomizu and Pinkall [4] states that locally M is affine equivalent to the graph immersion of a certain function F.

For any vector fields X, Y tangent to a graph hypersurface M, one can decompose $D_X f_*(Y)$ into its tangential and transverse components:

$$D_X f_*(Y) = f_*(\nabla_X Y) + h(X,Y)\xi, \tag{1.1}$$

where D is the canonical flat connection on $\mathbf{R}^{n+1}$, h is a symmetric tensor of type $(0,2)$ and ∇ is the induced affine connection.

If h is non-degenerate, h defines a semi-Riemannian metric on M which is called the affine metric of M.

Let $\hat{\nabla}$ be the Levi-Civita connection of (M,h) and K the difference tensor $\nabla - \hat{\nabla}$ on M. By taking the trace of K, one obtains a so-called Tchebychev form $T(X) := (1/n)\,\mathrm{trace}\{Y \to K(X,Y)\}$. The Tchebychev vector field $T^{\#}$ can then be defined by $h(T^{\#}, X) = T(X)$.

As usual we assume that h is definite. In case that h is negative definite, we shall replace ξ by $-\xi$ for the affine normal. In this way, the symmetric $(0,2)$-tensor h is always positive definite and thus always defines a Riemannian metric on M.

In this article we prove a general inequality for graph hypersurfaces in $\mathbf{R}^{n+1}$. We also classify the extremal class of graph hypersurfaces which satisfy the equality case of the optimal inequality identically.

2. Preliminaries. We recall some basic facts about graph hypersurfaces (for details see Nomizu and Sasaki's book [5]).

Let $f: M \to \mathbf{R}^{n+1}$ be a graph hypersurface. Then the equations of Gauss and Codazzi are given respectively by

$$R(X,Y)Z = 0, \tag{2.1}$$

$$(\nabla_X h)(Y,Z) = (\nabla_Y h)(X,Z). \tag{2.2}$$

Denote by $\hat{\nabla}$ the Levi-Civita connection of h and by $\hat{K}$ and $\hat{R}$ the sectional curvature and the curvature tensor of $\tilde{\nabla}$, respectively.

The difference tensor K is defined by

$$K_X Y = K(X,Y) = \nabla_X Y - \hat{\nabla}_X Y, \tag{2.3}$$

which is a symmetric $(1,2)$-tensor field.

For each X, K_X is self-adjoint. The Tchebychev form T and the Tchebychev vector field $T^{\#}$ are defined by

$$T(X) = \frac{1}{n}\,\mathrm{trace}\, K_X, \tag{2.4}$$

$$h(T^{\#}, X) = T(X). \tag{2.5}$$

It is well-known that for graph hypersurfaces we have

$$h(K_X Y, Z) = h(Y, K_X Z), \tag{2.6}$$

$$\hat{R}(X,Y)Z = K_Y K_X Z - K_X K_Y Z, \tag{2.7}$$

$$(\hat{\nabla}_X K)(Y,Z) = (\hat{\nabla}_Y K)(Z,X) = (\hat{\nabla}_Z K)(X,Y). \tag{2.8}$$

3. A general optimal inequality. For any 2-plane section π at $p \in M$, let $\hat{K}(\pi)$ denote the sec-

2000 Mathematics Subject Classification. Primary 53A15; Secondary 53C40, 53B20, 53B25.

tional curvature of (M,h) associated with π. The scalar curvature $\hat{\tau}$ of (M,h) at p is defined to be $\hat{\tau}(p) = \sum_{i<j} \hat{K}(e_i \wedge e_j)$, where $e_1,\dots,e_n$ is a h-orthonormal basis of T_pM.

For graph hypersurfaces we have the following general inequality.

Theorem 1. *If M is a definite graph hypersurface in $\mathbf{R}^{n+1}$, $n \geq 2$, then the Tchebychev vector field satisfies*

$$\hat{\tau} \geq \frac{n^2(1-n)}{2(n+2)} h(T^\#, T^\#). \tag{3.1}$$

The equality case of inequality (3.1) *holds at a point $p \in M$ if and only if we have*

$$\begin{aligned} &K(e_1,e_1) = 3\lambda e_1,\ K(e_1,e_j) = \lambda e_j, \\ &K(e_i,e_j) = 0,\ K(e_j,e_j) = \lambda e_1, \\ &\quad 2 \leq i \neq j \leq n \end{aligned} \tag{3.2}$$

with respect to some suitable h-orthonormal basis $e_1,\dots,e_n$ of T_pM.

Proof. Let M be a definite graph hypersurface in $\mathbf{R}^{n+1}$ and let $e_1,\dots,e_n$ be a h-orthonormal basis. We put $K^i_{jk} = h(K(e_j,e_k),e_i)$. From (2.6) we have

$$K^i_{jk} = K^j_{ik} = K^k_{ij}, \quad i,j,k = 1,\dots,n. \tag{3.3}$$

From the definition of Tchebychev vector we find

$$\begin{aligned} &n^2 h(T^\#, T^\#) \\ &\quad = \sum_i \Big(\sum_j (K^i_{jj})^2 + 2\sum_{j<k} K^i_{jj} K^i_{kk} \Big). \end{aligned} \tag{3.4}$$

By applying equation (2.7) we have

$$2\hat{\tau} = h(K,K) - n^2 h(T^\#, T^\#). \tag{3.5}$$

Thus, by (3.3), (3.4) and (3.5), we obtain

$$\begin{aligned} 2\hat{\tau} &= 2\sum_{i\neq j}(K^i_{jj})^2 + 6\sum_{i<j<k}(K^i_{jk})^2 \\ &\quad - \sum_i \sum_{j\neq k} K^i_{jj} K^i_{kk}. \end{aligned} \tag{3.6}$$

From (3.4) and (3.6) we find

$$\begin{aligned} &n^2 h(T^\#,T^\#) + \frac{2(n+2)}{n-1}\hat{\tau} \\ &= \sum_i (K^i_{ii})^2 + \frac{3n+5}{n-1}\sum_{i\neq j}(K^i_{jj})^2 \\ &\quad + \frac{6(n+2)}{n-1}\sum_{i<j<k}(K^i_{jk})^2 - \frac{3}{n-1}\sum_i\sum_{j\neq k} K^i_{jj}K^i_{kk} \\ &= \sum_i (K^i_{ii})^2 + \frac{6(n+2)}{n-1}\sum_{i<j<k}(K^i_{jk})^2 \\ &\quad - \frac{6}{n-1}\sum_{j\neq i} K^i_{ii}K^i_{jj} + \frac{9}{n-1}\sum_{j\neq i}(K^i_{jj})^2 \\ &\quad + \frac{3}{n-1}\sum_{i\neq j,k}\sum_{j<k}(K^i_{jj} - K^i_{kk})^2 \\ &= \frac{6(n+2)}{n-1}\sum_{i<j<k}(K^i_{jk})^2 \\ &\quad + \frac{1}{n-1}\sum_{j\neq i}\left(K^i_{ii} - 3K^i_{jj}\right)^2 \\ &\quad + \frac{3}{n-1}\sum_{i\neq j,k}\sum_{j<k}(K^i_{jj} - K^i_{kk})^2 \\ &\geq 0 \end{aligned}$$

which implies (3.1).

It is easy to see that the equality sign of (3.1) holds if and only if $K^i_{ii} = 3K^i_{jj}$ and $K^i_{jk} = 0$ for distinct i, j, k. Thus, if we choose $e_1,\dots,e_n$ in such way that e_1 is parallel to the Tchebychev vector field $T^\#$, we obtain (3.2).

The converse is easy to verify. □

4. The equality case.

Theorem 2. *If $f : M \to \mathbf{R}^{n+1}$, $n \geq 2$, is a definite graph hypersurface satisfying the equality case of* (3.1) *identically, then M is affinely equivalent to an open part of one of the following hypersurfaces*:

(I) *The paraboloid defined by*

$$\left(u_1, u_2, \dots, u_n, \frac{1}{2}\sum_{j=1}^n u_j^2 \right).$$

(II) *The hypersurface defined by*

$$\left(u_2, \dots, u_n, \frac{s^4}{4} + \sum_{j=2}^n u_j^2, -\frac{s^2}{4} \right).$$

(III) *A hypersurface defined by*

$$\begin{aligned} &\frac{\left\{(\mathrm{ns}(2as,k) - \mathrm{ds}(2as,k))(\mathrm{ds}(2as,k) - \mathrm{cs}(2as,k))\right\}^{\frac{3}{2}}}{\mathrm{ns}(2as,k) + \mathrm{cs}(2as,k) - 2ds(2as,k)} \\ &\quad \times \left(\sin u_2, \dots, \sin u_n \prod_{j=2}^{n-1}\cos u_j, \prod_{j=2}^{n}\cos u_j, 0 \right) \\ &\quad - \left(0,\dots,0, \frac{1 + \mathrm{dn}(2as,k)}{a^2(1 + \mathrm{cn}(2as,k) + 2\mathrm{dn}(2as,k))} \right), \end{aligned}$$

where $k = 1/\sqrt{2}$ is the modulus of Jacobi's elliptic functions and a is an arbitrary positive number.

(IV) *A hypersurface defined by*

$$\mathrm{ds}(as,k)\big(\mathrm{cs}(as,k)-\mathrm{ns}(as,k)\big)\times\Bigg(\sin u_2,\sin u_3\cos u_2,\ldots,\sin u_n\prod_{j=2}^{n-1}\cos u_j,\prod_{j=2}^{n}\cos u_j,\mathrm{nd}(as,k)-\mathrm{cd}(as,k)\Bigg),$$

where $k=1/\sqrt{2}$ is the modulus of Jacobi's elliptic functions and a is an arbitrary positive number.

Proof. Let M be a definite graph hypersurface satisfying the equality case of (3.1) identically. Then we have (3.2) with respect to some h-orthonormal frame $\{e_1,\ldots,e_n\}$. Let $\omega^1,\ldots,\omega^n$ be the dual 1-forms of $e_1,\ldots,e_n$ with respect to h and (ω_i^j) the connection form on (M,h), so we have $\hat\nabla_X e_i=\sum_{j=1}^n\omega_i^j(X)e_j$.

Case (i): $\lambda=0$ identically. In this case, we have $K=0$, $\nabla=\hat\nabla$. Hence, M is affinely equivalent to an open portion of the paraboloid given in (I).

Case (ii): $\lambda\neq 0$. Let $U=\{p\in M: T^\#(p)\neq 0\}$. Then U is a nonempty open subset. From Theorem 1 we have $U=\{p\in M: K\neq 0 \text{ at } p\}$. By applying (2.8) and (3.2) we find

$$e_1\lambda=\lambda\omega_1^j(e_j),\ \ e_j\lambda=0,\ j=2,\ldots,n, \tag{4.1}$$

$$\omega_1^j(e_k)=\omega_1^j(e_1)=0,\quad 1<j\neq k\leq n. \tag{4.2}$$

From (4.1) and (4.2) we obtain

$$\omega_1^j=e_1(\ln\lambda)\omega^j,\quad j=2,\ldots,n. \tag{4.3}$$

Let $\mathcal{D}$ denote the distribution on U spanned by e_1 and $\mathcal{D}^\perp$ the h-orthogonal complementary distribution of $\mathcal{D}$ which is spanned by $\{e_2,\ldots,e_n\}$.

Lemma 1. *On U we have*:

(a) *The integral curves of e_1 are geodesics of (M,h).*

(b) *Distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ are both integrable.*

(c) *There exist a local coordinate system $\{x_1,\ldots,x_n\}$ such that*

(c.1) *$\mathcal{D}$ is spanned by $\{\partial/\partial x_1\}$ and $\mathcal{D}^\perp$ is spanned by $\{\partial/\partial x_2,\ldots,\partial/\partial x_n\}$*;

(c.2) *$e_1=\partial/\partial x_1$, $\omega^1=dx_1$ and h takes the form*: $h=dx_1^2+\sum_{j,k=2}^n h_{jk}dx_jdx_k$.

(d) *λ is a function of $s:=x_1$ satisfying*

$$\frac{d^2\lambda}{ds^2}=2\lambda^3. \tag{4.4}$$

Proof of Lemma 1. From (4.2) and (4.3), we find $\hat\nabla_{e_1}e_1=d\omega^1=0$, which implies that the integral curves of e_1 are h-geodesic.

By using (4.2) we get $h([e_j,e_k],e_1)=0$ which implies that $\mathcal{D}^\perp$ is integrable. Also, since $\mathcal{D}$ is of rank one, $\mathcal{D}$ is trivially integrable.

Because $\mathcal{D}$ is of rank one, there exists a local coordinate system $\{y_1,\ldots,y_n\}$ such that $e_1=\partial/\partial y_1$. Since $\mathcal{D}^\perp$ is integrable too, there also exists a local coordinate system $\{z_1,\ldots,z_n\}$ such that $\mathcal{D}^\perp$ is spanned by $\partial/\partial z_2,\ldots,\partial/\partial z_n$. Hence, if we put $x_1=y_1,x_2=z_2,\ldots,x_n=z_n$, then $\{x_1,\ldots,x_n\}$ is a local coordinate system which satisfies conditions (c.1) and (c.2).

Statement (c) and (4.1) imply that λ depends only on s. Using (2.7) and (3.2) we get

$$h(\hat R(e_1,e_2)e_2,e_1)=-2\lambda^2. \tag{4.5}$$

On the other hand, (4.1), (4.2) and (4.3) imply that

$$h(\hat R(e_1,e_2)e_2,e_1)=-(\ln\lambda)''-(\ln\lambda)'^2. \tag{4.6}$$

Combining these two equations yields (4.4). □

Lemma 2. *Up to sign and translation on s, the non-trivial solutions of differential equation* (4.4) *are the following functions*:

(a) $\lambda=s^{-1}$,

(b) $\lambda=a\,\mathrm{ns}(2as,1/\sqrt{2})+a\,\mathrm{cs}(2as,1/\sqrt{2})$, $a>0$,

(c) $\lambda=a\,\mathrm{ds}(as,1/\sqrt{2})$, $a>0$.

Proof of Lemma 2. Clearly, differential equation (4.4) admits no non-trivial constant solution. So, we may assume λ is non-constant. Hence (4.4) yields $\lambda' d\lambda'=2\lambda^3 d\lambda$ which implies that

$$\pm(s+b)=\int^\lambda\frac{dt}{\sqrt{t^4+c}}, \tag{4.7}$$

for some constants b, c.

Case (1): $c=0$. In this case, (4.7) yields $\pm(s+c_2)=1/\lambda$ which gives solution (a).

Case (2): $c>0$. If we put $c=a^4$ for a positive number a, we obtain solution (b) from (4.7).

Case (3): $c<0$. If we put $c=-a^4/4$, then we obtain solution (c). □

Lemma 3. *The distribution $\mathcal{D}$ is auto-parallel and its h-orthogonal complementary distribution $\mathcal{D}^\perp$ is spherical on U. Moreover, when $n\geq 3$, each leave of $\mathcal{D}^\perp$ in (M,h) is of constant curvature $\lambda'^2/\lambda^2-\lambda^2$.*

Proof of Lemma 3. First, it is easy to see that Lemma 1 implies that $\mathcal{D}$ is auto-parallel.

Let X,Y be any two vector fields in $\mathcal{D}^\perp$ and e_1 a h-unit vector field in $\mathcal{D}$. Then (2.8), (3.2) and the auto-parallelism of $\mathcal{D}$ imply that

$$
\begin{aligned}
-3\lambda\, h(\hat\nabla_X Y, e_1) &= -h(\hat\nabla_X Y, K(e_1, e_1))\\
&= h(Y, (\hat\nabla K)(X, e_1, e_1)) + 2h(Y, K(e_1, \hat\nabla_X e_1))\\
&= h(Y, (\hat\nabla K)(e_1, e_1, X)) + 2\lambda\, h(Y, \hat\nabla_X e_1)\\
&= h(Y, \nabla_{e_1} K(e_1, X)) - h(Y, K(e_1, \hat\nabla_{e_1} X))\\
&\quad + 2\lambda\, h(Y, \hat\nabla_X e_1)\\
&= h(Y, \hat\nabla_{e_1}(\lambda X)) - h(Y, K(e_1, \hat\nabla_{e_1} X))\\
&\quad + 2\lambda\, h(Y, \hat\nabla_X e_1)\\
&= (e_1\lambda) h(X, Y) + 2\lambda\, h(Y, \hat\nabla_X e_1)\\
&= (e_1\lambda) h(X, Y) - 2\lambda\, h(\hat\nabla_X Y, e_1).
\end{aligned}
$$

Thus we obtain

$$h(\hat\nabla_X Y, e_1) = -(e_1 \ln\lambda) h(X, Y), \tag{4.8}$$

which shows that the leaves of $\mathcal{D}^\perp$ are totally umbilical hypersurfaces with constant mean curvature. Since the codimension of leaves is one, the distribution $\mathcal{D}^\perp$ is a spherical distribution.

When $n \geq 3$, it follows from (2.7), (3.2) and (4.8) that each leave of $\mathcal{D}^\perp$ is of constant curvature $\lambda'^2/\lambda^2 - \lambda^2$ with respect to the metric induced from (M, h). This proves Lemma 3. □

Now, let us assume that $n \geq 3$. It follows from Lemma 3 and a result of [3] that (U, h) is locally the warped product $I \times_{\varphi(s)} N(\bar c)$ of an open interval I and a Riemannian $(n-1)$-manifold $N(\bar c)$ of constant curvature $\bar c$ with a suitable warping function φ. So, the metric h on $I \times_\varphi N$ is given by

$$h = ds^2 + \varphi^2 \tilde h, \tag{4.9}$$

where $\tilde h$ is the constant curvature metric on $N(\bar c)$. Without loss of generality, we may choose $\bar c = 1$, 0, or -1 according to $c > 0$, $c = 0$, or $c < 0$. Obviously, vectors tangent to first factor I are in $\mathcal{D}$ and vectors tangent to the second factor $N(\bar c)$ are in $\mathcal{D}^\perp$.

By applying Theorem 1, equation (2.8) of Codazzi and (4.9), we find $(\ln\lambda)_s = (\ln\varphi)_s$. Hence, we have

$$\varphi = \alpha\lambda \tag{4.10}$$

for some nonzero real number α.

It is well-known that the curvature tensors $\hat R$ and R^F of M and N for the warped product $M := I \times_\varphi N$ are related by

$$
\begin{aligned}
\hat R(X, Y)Z &= R^F(X, Y)Z\\
&\quad - \left(\frac{\varphi'}{\varphi}\right)^2 (h(Y, Z)X - h(X, Z)Y)
\end{aligned}
\tag{4.11}
$$

for vector fields X, Y, Z tangent to N. From (4.10) and (4.11) we have

$$\hat K\left(\frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3}\right) = \frac{\bar c - \varphi'^2}{\varphi^2} = \frac{c - \lambda'^2}{\lambda^2}, \quad c = \frac{\bar c}{\alpha^2}. \tag{4.12}$$

On the other hand, Theorem 1 and (2.7) give

$$K\left(\frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3}\right) = -\lambda^2. \tag{4.13}$$

Combining (4.12) and (4.13) shows that λ satisfies

$$\lambda'^2 - \lambda^4 = c. \tag{4.14}$$

Since the constant c is equal to zero, a positive number, or a negative number, according to λ is given by cases (a), (b), or (c) of Lemma 2, the open subset U is the whole hypersurface M in cases (b) and (c); and U is dense in M if λ is given by case (a).

Now, we divide the proof into three cases.

Case (a): $\lambda = s^{-1}$. In this case, we have $c = 0$. Thus (M, h) is locally the warped product $\mathbf{R} \times_{\varphi(s)} \mathbf{E}^{n-1}$ with a suitable warping function φ. So, with respect to a natural Euclidean coordinate system $\{u_2, \dots, u_n\}$, the warped product metric is

$$
\begin{aligned}
h &= ds^2 + \varphi^2(s) h_0,\\
h_0 &= du_2^2 + du_3^2 + \cdots + du_n^2.
\end{aligned}
\tag{4.15}
$$

It follows from (4.15) that the sectional curvature of $\mathbf{R} \times_{\varphi(s)} \mathbf{E}^{n-1}$ satisfies

$$\hat K\left(\frac{\partial}{\partial s} \wedge \frac{\partial}{\partial u_k}\right) = -\frac{\varphi''(s)}{\varphi(s)}. \tag{4.16}$$

On the other hand, (2.7), (3.2) and $\lambda = s^{-1}$ give

$$\hat K\left(\frac{\partial}{\partial s} \wedge \frac{\partial}{\partial u_k}\right) = -\frac{2}{s^2}. \tag{4.17}$$

Hence, by combining (4.16) and (4.17), we obtain

$$s^2\varphi''(s) = 2\varphi(s), \tag{4.18}$$

which gives $\varphi = k_1 s^2 + k_2 s^{-1}$ for some constant k_1, k_2 not both zero. So, (4.15) becomes

$$h = ds^2 + \left(k_1 s^2 + \frac{k_2}{s}\right)^2 h_0 \tag{4.19}$$

which yields

$$
\begin{aligned}
&\hat\nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial s} = \hat\nabla_{\frac{\partial}{\partial u_i}} \frac{\partial}{\partial u_j} = 0,\\
&\hat\nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial u_i} = \frac{2k_1 s^3 - k_2}{s(k_1 s^3 + k_2)} \frac{\partial}{\partial u_i},
\end{aligned}
\tag{4.20}
$$

$$\hat{\nabla}_{\frac{\partial}{\partial u_i}} \frac{\partial}{\partial u_i} = \frac{(k_1 s^3 + k_2)(k_2 - 2k_1 s^3)}{s^3} \frac{\partial}{\partial s},$$

for $2 \leq i \neq j \leq n$. By combining (1.1), (2.3), (3.2), (4.19) and (4.20) we know that the immersion $f : M \to \mathbf{R}^{n+1}$ satisfies

$$(4.21) \qquad \begin{aligned} f_{ss} &= \frac{3}{s} f_s + \xi, \\ f_{su_j} &= \frac{3k_1 s^2}{k_1 s^3 + k_2} f_{u_j}, \\ f_{u_j u_j} &= \frac{(k_1 s^3 + k_2)(2k_2 - k_1 s^3)}{s^3} f_s \\ &\quad + \left(k_1 s^2 + \frac{k_2}{s}\right)^2 \xi, \\ f_{u_i u_j} &= 0, \quad 2 \leq i \neq j \leq n. \end{aligned}$$

From the condition $(f_{ss})_{u_j} = (f_{su_j})_s$ we find $k_1 = 0$. For simplicity, we may assume $k_2 = 1$; thus $\varphi = s^{-1}$. After solving the partial differential system (4.21) with $k_1 = 0, k_2 = 1$, we obtain

$$f(s, u_1, \ldots, u_n) = \sum_{j=2}^{n} c_j u_j + b \sum_{j=2}^{n} u_j^2 + \frac{b}{4} s^4 - \frac{s^2}{4} \xi$$

for some basis $b, c_2, \ldots, c_n$. Therefore, we conclude that M is affinely equivalent to the hypersurface defined by (II).

Case (b): $\lambda = a\,\mathrm{ns}(2as, k) + a\,\mathrm{cs}(2as, k)$ with $a > 0$, $k = 1/\sqrt{2}$. In this case, we have $c = a^4$ which implies $\bar{c} = 1$ and $\alpha = 1/a^2$ by (4.12). Furthermore, by (4.10), we know that (M, h) is locally the warped product of an open interval and a unit $(n-1)$-sphere with warped product metric:

$$h = ds^2 + \frac{\{\mathrm{ns}\,(2as, k) + \mathrm{cs}\,(2as, k)\}^2}{a^2} h_1,$$

$$h_1 = dx_2^2 + \cos^2 x_2 dx_3^2 + \cdots + \prod_{j=2}^{n-1} \cos^2 x_j dx_n^2.$$

Thus the Levi-Civita connection satisfies

$$\hat{\nabla}_{\frac{\partial}{\partial s}} \frac{\partial}{\partial s} = 0, \quad \hat{\nabla}_{\frac{\partial}{\partial s}} \frac{\partial}{\partial u_i} = -2a\mathrm{ds}(2as, k) \frac{\partial}{\partial u_i},$$

$$\hat{\nabla}_{\frac{\partial}{\partial u_i}} \frac{\partial}{\partial u_j} = -\tan u_i \frac{\partial}{\partial u_j},$$

$$\hat{\nabla}_{\frac{\partial}{\partial u_2}} \frac{\partial}{\partial u_2} = \frac{2}{a} \mathrm{ds}(2as, k)(\mathrm{ns}(2as, k) + \mathrm{cs}(2as, k))^2 \frac{\partial}{\partial s},$$

$$\hat{\nabla}_{\frac{\partial}{\partial u_i}} \frac{\partial}{\partial u_i} = \frac{2}{a} \mathrm{ds}(2as, k)(\mathrm{ns}(2as, k) + \mathrm{cs}(2as, k))^2$$
$$\times \prod_{i=2}^{j-1} \cos^2 u_i \frac{\partial}{\partial s} + \sum_{k=2}^{j-1} \left(\frac{\sin 2u_k}{2} \prod_{l=k+1}^{j-1} \cos^2 u_l \right) \frac{\partial}{\partial u_k}$$

for $2 \leq i \neq j \leq n$.

Hence the immersion f satisfies the following differential system

$$\begin{aligned} f_{ss} &= 3a\{\mathrm{ns}\,(a2s, k) + \mathrm{cs}\,(2as, k)\} f_s + \xi, \\ f_{su_k} &= a\{\mathrm{ns}(2as, k) + \mathrm{cs}(2as, k) - 2\,\mathrm{ds}(2as, k)\} f_{u_k}, \\ &\qquad k = 2, \ldots, n, \\ f_{u_i u_j} &= -\tan x_i f_{u_j}, \quad 2 \leq i < j \leq n, \\ f_{u_j u_j} &= \frac{(\mathrm{ns}\,(2as, k) + \mathrm{cs}\,(2as, k))^2}{a} \times \\ &(\mathrm{ns}\,(2as, k) + \mathrm{cs}\,(2as, k) + 2\,\mathrm{ds}(2as, k)) \prod_{i=2}^{j-1} \cos^2 u_i f_s \\ &+ \sum_{k=2}^{j-1} \left(\frac{\sin 2u_k}{2} \prod_{l=k+1}^{j-1} \cos^2 u_l \right) f_{u_k} \\ &+ \frac{(\mathrm{ns}\,(2as, k) + \mathrm{cs}\,(2as, k))^2}{a^2} \left(\prod_{i=2}^{j-1} \cos^2 u_i \right) \xi, \\ &\qquad j = 2, \ldots, n. \end{aligned}$$

Solving the first equation of this system gives

$$(4.22) \quad \begin{aligned} f &= A - \frac{1 + \mathrm{dn}(2as, k)}{a^2(1 + \mathrm{cn}(2as, k) + 2\mathrm{dn}(2as, k))} \xi \\ &+ \frac{\{(\mathrm{ns}(2as, k) - \mathrm{ds}(2as, k))(\mathrm{ds}(2as, k) - \mathrm{cs}(2as, k))\}^{\frac{3}{2}} B}{a(\mathrm{ns}(2as, k) + \mathrm{cs}(2as, k) - 2ds(2as, k))} \end{aligned}$$

for $A = A(u_2, \ldots, u_n)$ and $B = B(u_2, \ldots, u_n)$. By substituting (4.22) into the second equation of the system we obtain $A_{u_k} = 0$ which implies that A is a constant vector, say c_0. Finally, by substituting (4.22) with $A = c_0$ into the remaining equations of the system, we find

$$B = c_1 \sin u_2 + c_2 \sin u_3 \cos u_2 + \cdots + c_{n-1} \sin u_n \prod_{j=2}^{n-1} \cos u_j + c_n \prod_{j=2}^{n} \cos u_j.$$

Consequently, the hypersurface is affinely equivalent to a hypersurface given by (III).

Case (c): $\lambda = a\,\mathrm{ds}\,(as, k)$ with $a > 0, k = 1/\sqrt{2}$. In this case, we obtain $4c = -a^4 < 0$ which yields $\bar{c} = -1$, $\alpha = 2/a^2$. Thus (M, h) is locally the warped product of the real line and a unit hyperbolic $(n-1)$-space $H^{n-1}(-1)$ with warped product metric:

$$(4.23) \qquad h = ds^2 + \frac{4}{a^2} \mathrm{ds}^2\,(as, k)\, h_{-1}$$

with

$$h_{-1} = dx_2^2 + \cosh^2 x_2 dx_3^2 + \cdots + \prod_{j=2}^{n-1} \cosh^2 x_j dx_n^2.$$

Thus we have

$$\hat\nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial s} = 0, \quad \hat\nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial u_i} = -a\,\mathrm{cd}(as,k)\mathrm{ns}(as,k)\frac{\partial}{\partial u_i},$$

$$\hat\nabla_{\frac{\partial}{\partial u_i}} \frac{\partial}{\partial u_j} = \tanh u_i \frac{\partial}{\partial u_j}, \quad 2 \le i < j \le n,$$

$$\hat\nabla_{\frac{\partial}{\partial u_2}} \frac{\partial}{\partial u_2} = \frac{4}{a}\mathrm{cs}(as,k)\mathrm{ds}(as,k)\mathrm{ns}(as,k)\frac{\partial}{\partial s},$$

$$\hat\nabla_{\frac{\partial}{\partial u_i}} \frac{\partial}{\partial u_i} = \frac{4}{a}\mathrm{cs}(as,k)\mathrm{ds}(as,k)\mathrm{ns}(as,k) \times \prod_{i=2}^{j-1} \cosh^2 u_i \frac{\partial}{\partial s} - \sum_{k=2}^{j-1}\left(\frac{\sinh 2u_k}{2}\prod_{l=k+1}^{j-1}\cosh^2 u_l\right)\frac{\partial}{\partial u_k},$$
$$2 \le i \ne j \le n.$$

From these we know that the immersion f satisfies the following differential system:

$$f_{ss} = 3a\,\mathrm{ds}(as,k) f_s + \xi,$$

$$f_{su_j} = a\big(\mathrm{ds}(as,k) - \mathrm{cd}(as,k)\,\mathrm{ns}(as,k)\big) f_{u_j}, \quad j = 2,\ldots,n,$$

$$f_{u_2u_2} = \frac{4}{a^2}\mathrm{ds}^2(as,k)\xi + \frac{4}{a}\mathrm{ds}(as,k)\{\mathrm{ds}^2(as,k) + \mathrm{cs}(as,k)\,\mathrm{ns}(as,k)\} f_s,$$

$$f_{u_ju_j} = \frac{4}{a^2}\mathrm{ds}^2(as,k)\prod_{i=2}^{j-1}\cosh^2 u_i \xi + \frac{4}{a}\mathrm{ds}(as,k)\{\mathrm{ds}^2(as,k) + \mathrm{cs}(as,k)\,\mathrm{ns}(as,k)\} \times \prod_{i=2}^{j-1}\cosh^2 u_i f_s - \sum_{\ell=2}^{j-1}\left(\frac{\sinh 2u_\ell}{2}\prod_{i=\ell+1}^{j-1}\cosh^2 u_i\right) f_{u_\ell}$$

$$f_{u_iu_\ell} = \tanh u_i f_{u_\ell}, \quad 2 \le i < \ell \le n; \quad j > 2.$$

Solving the first equation of the last differential system gives

$$f = A + \mathrm{ds}(as,k)\big(\mathrm{cs}(as,k) - \mathrm{ns}(as,k)\big)B - \frac{\big(\mathrm{ns}(as,k) - \mathrm{cs}(as,k)\big)^2}{a^2}\xi \tag{4.24}$$

for some functions $A = A(u_2,\ldots,u_n)$ and $B = B(u_2,\ldots,u_n)$. Substituting (4.24) into the second equation of the system yields $A_{u_k} = 0$. Thus, A is a constant vector. Finally, by substituting (4.24) into the remaining equations of the system, we conclude that the hypersurface is affinely equivalent to a hypersurface given by (IV).

When $n = 2$, Hiepko's result implies that M is locally the warped product of an open interval I and the real line $\mathbf{R}$ with warped product metric $g = dx^2 + \varphi^2 dy^2$. Using (2.8) we have $\varphi = \alpha\lambda$ for some nonzero constant α. Because λ is given by one of the three functions given in Lemma 2, the same arguments as for $n \ge 3$ yield the same results for $n = 2$ as well. □

Remark 1. For the corresponding general optimal inequalities of affine hypersurfaces in centroaffine geometry, see [2].

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