Canonical curves of genus eight

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(Communicated by Shigefumi Mori, M. J. A., March 12, 2003)

Abstract: A non-tetragonal curve of genus 8 is a complete intersection of divisors in either $\mathbf{P}^2 \times \mathbf{P}^2$, a 6-dimensional weighted Grassmannian or the 8-dimensional Grassmannian.

Key words: Canonical curve; gonality; Grassmann variety.

Let $C = C_{14} \subset \mathbf{P}^7$ be a canonical curve of genus 8 over an algebraically closed field k. If C has no g^2_7, then $C_{14} \subset \mathbf{P}^7$ is a transversal linear section $[G(2,6) \subset \mathbf{P}^{14}] \cap H_1 \cap \cdots \cap H_7$ of the 8-dimensional Grassmannian ([M2]). This is the case $\langle 8 \rangle$ of the Flowchart below. In this article we study the case where C has a g^2_7 α. The system of defining equations of C_{14} is easily found from the following: ([M1] Prop. 5)

Theorem. (i) *Assume that C has no g^1_4. If $\alpha^2 \cong K_C$, then C is the complete intersection of the 6-dimensional weighted Grassmannian w-$G(2,5) \subset \mathbf{P}(1^3 : 2^6 : 3)$ with a subspace $\mathbf{P}(11122)$, where $w = (1,1,1,3,3)/2$ (Case $\langle 7 \rangle$). Otherwise C is the complete intersection of three divisors of bidegree $(1,1)$, $(1,2)$ and $(2,1)$ in $\mathbf{P}^2 \times \mathbf{P}^2$ (Case $\langle 6 \rangle$).*

(ii) *Assume that C has a g^1_4 but no g^2_6. Then C is the complete intersection of four divisors of bidegree $(1,1)$, $(1,1)$, $(0,2)$ and $(1,2)$ in $\mathbf{P}^1 \times \mathbf{P}^4$ (Case $\langle 5 \rangle$).*

Here a g^r_d is a line bundle of degree d and $h^0 \geq r+1$.

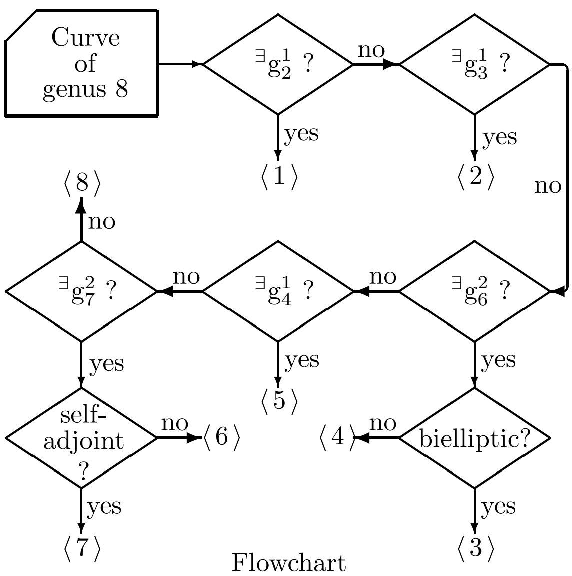
Corollary. *C is a complete intersection of divisors in a variety X which is either a non-singular toric variety or a weighted Grassmannian*:

Case	$\langle 1 \rangle$	$\langle 2 \rangle$	$\langle 3 \rangle$	$\langle 4 \rangle$	
X	$\mathbf{F}_9$	$\mathbf{P}^1 \times \mathbf{P}^1, \mathbf{F}_2$	W'_7	$Bl_p\ \mathbf{P}^3$	
Cliff C	0	1			2

$\langle 5 \rangle$	$\langle 6 \rangle$	$\langle 7 \rangle$	$\langle 8 \rangle$
$\mathbf{P}^1 \times \mathbf{P}^4$	$\mathbf{P}^2 \times \mathbf{P}^2$	w-$G(2,5)$	$G(2,6)$
	3		

Here W'_7 is the $\mathbf{P}^1$-bundle $\mathbf{P}(\mathcal{O}_S \oplus \mathcal{O}_S(-K))$ over S_7, the blow-up of $\mathbf{P}^2$ at two points. The bottom row indicates the Clifford index of C.

This is applied to the K3-extension problem in [I].



Flowchart

1. Cases of small Clifford index. The cases $\langle 1 \rangle, \ldots, \langle 4 \rangle$ are easy.

Case $\langle 1 \rangle$. C is a double covering $y^2 = f_{18}(x_0, x_1)$ of $\mathbf{P}^1$ in the weighted projective space $\mathbf{P}(1:1:9)$, whose minimal resolution $\mathbf{F}_9$ is the toric variety X.

Case $\langle 2 \rangle$. X, a 2-dimensional rational scroll of degree 6, is the quadric hull of $C_{14} \subset \mathbf{P}^7$ ([ACGH] III §3).

Case $\langle 3 \rangle$. $C_{14} \subset \mathbf{P}^7$ is contained in the cone over an elliptic curve $E_7 \subset \mathbf{P}^6$ of degree 7. E_7 is a hyperplane section of a smooth del Pezzo surface $S_7 \subset \mathbf{P}^7$ of degree 7. Let B be the branch locus of the double covering $C \to E_7$. Then there exists a member $D \in |-2K_S|$ with $D \cap C = B$. In the

2000 Mathematics Subject Classification. Primary 14H45.

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$\mathbf{P}^1$-bundle W_7', C is the intersection of the double covering of S with branch D and the inverse image of E_7.

Case $\langle 4 \rangle$. Let α be a non-bielliptic g^2_6, and β its Serre adjoint $K_C\alpha^{-1}$, which is a g^3_8 by Riemann-Roch. Then both α and β are base point free. $\Phi_{|\alpha|}$ is a birational morphism onto a plane sextic C_6 with two nodes, one of which may be infinitely near. Let $\pi : S \xrightarrow{\pi_2} S' \xrightarrow{\pi_1} \mathbf{P}^2$ be the composite of the blowing-ups at these nodes, h the pull-back of a line, e_1 and e_2 the total transform of the exceptional divisors. Since $-K_S \sim 3h - e_1 - e_2$ and $C \sim 6h - 2e_1 - 2e_2$, we have $\alpha = h|_C$, $\beta = (2h - e_1 - e_2)|_C$ and $\beta\alpha^{-1} = (h - e_1 - e_2)|_C$.

The morphism $\Phi_{|\beta|}$ is birational onto a space curve $C_8 \subset \mathbf{P}^3$ of degree 8. $\Phi_{|\beta|}$ extends to the morphism $f = \Phi_{|2h-e_1-e_2|} : S \to \mathbf{P}^3$ onto a quadric surface Q. f contracts the strict transform $L \in |h - e_1 - e_2|$ of the line passing through the two nodes of C_6 to a nonsingular point p of Q. Since $L.C = 2$, p is a double point of C_8. C_8 is a complete intersection of Q and a quartic surface since $C + 2L \sim 4(2h - e_1 - e_2)$. C itself is the complete intersection of two divisors in the blow-up X of $\mathbf{P}^3$ at p. One is the strict transform $\simeq S$ of Q and the other belongs to $|-K|$.

Case $\langle 5 \rangle$. Let α be a g^1_4 and β its Serre adjoint. Then $|\alpha|$ is base point free since C is not trigonal. β is a g^4_{10} by Riemann-Roch and very ample by assumption since C has no g^2_6 or g^3_8.

Lemma 1.1. *The multiplication map* $\mu : H^0(\alpha) \otimes H^0(\beta) \longrightarrow H^0(K_C)$ *is surjective.*

Proof. By the base point free pencil trick ([ACGH]), the kernel of μ is $H^0(\alpha^{-1}\beta)$. If μ is not surjective, then $\alpha^{-1}\beta$ is a g^2_6. This is a contradiction. □

There is a commutative diagram of embeddings

$$\begin{array}{ccc} \mathbf{P}^1 \times \mathbf{P}^4 & \xrightarrow{\text{Segre}} & \mathbf{P}^9 = \mathbf{P}^*(H^0(\alpha) \otimes H^0(\beta)) \\ {\scriptstyle \Phi_{|\alpha|} \times \Phi_{|\beta|}} \uparrow & & \uparrow {\scriptstyle \mu^*} \\ C & \xrightarrow{\text{canonical}} & \mathbf{P}^7 = \mathbf{P}^* H^0(\omega_C), \end{array}$$

where μ^* is the linear embedding associated with the surjection μ. By the lemma, the number of linearly independent $(1,1)$-forms vanishing on C is equal to 2. Therefore, C is contained in the intersection Y of two divisors of bidegree $(1,1)$ in $X = \mathbf{P}^1 \times \mathbf{P}^4$. Since every divisor of bidegree $(1,1)$ containing C is smooth, Y is smooth of dimension 3. Moreover $\operatorname{Pic} Y \cong \mathbf{Z}^2$ by Lefschetz theorem. By easy dimension count, there exists a divisor of bidegree $(1,2)$ and $(0,2)$ on Y which contain C. Since the degree of the complete intersection $Y \cap (1,2) \cap (0,1)$ is

$$(a+b)^2.(a+2b).(2b).(a+b) = 14ab^3 = 14 = \deg C,$$

it coincides with C, where $a = pr_1^*\mathcal{O}_{\mathbf{P}^1}(1)$ and $b = pr_2^*\mathcal{O}_{\mathbf{P}^4}(1)$.

2. Linear net of degree 7. Assume that C has a g^2_7 α but no g^1_4. Let $\overline{C} \subset \mathbf{P}^2$ be the image of the morphism $\Phi_{|\alpha|}$. Then $\overline{C}$ is of degree 7 and has no triple points. By the genus formula, $\overline{C}$ has 7 double points, some of which may be infinitely near. Therefore, there is a composition π of seven one-point-blowing-ups

$$S := S_{(7)} \longrightarrow S_{(6)} \longrightarrow \cdots \longrightarrow S_{(1)} \longrightarrow S_{(0)} = \mathbf{P}^2$$

such that $\Phi_{|\alpha|} : C \to \mathbf{P}^2$ lifts to $C \to S$. Let $E_i \subset S$, $1 \le i \le 7$, be the total transform of the exceptional divisor of the blow-up $S_{(i)} \longrightarrow S_{(i-1)}$ and h the pull back of a line. Then $C \subset S$ belongs to the linear system $|7h - 2\sum_{i=1}^7 E_i|$. Since the canonical class K_S of S is $-3h + \sum_{i=1}^7 E_i$,

$$H^i\Big(\mathcal{O}_S\Big((n-7)h + \sum_{i=1}^7 E_i\Big)\Big)$$

is the dual of $H^{2-i}(\mathcal{O}_S((4-n)h))$. Hence we have

Lemma 2.1. *The restriction map*

$$H^0\Big(S, \mathcal{O}_S\Big(nh - \sum_{i=1}^7 E_i\Big)\Big) \longrightarrow H^0\Big(C, \mathcal{O}_C\Big(nh - \sum_{i=1}^7 E_i\Big)\Big)$$

is surjective for every n. *Moreover, it is an isomorphism for* $n \le 6$.

By the adjunction formula

$$K_C = (K_S + C)|_C = h|_C + \Big(3h - \sum_{i=1}^7 E_i\Big)|_C,$$

the Serre adjoint $\beta = K_C\alpha^{-1}$ is isomorphic to $\mathcal{O}_C(3h - \sum_{i=1}^7 E_i)$. By Lemma 2.1, α is *self adjoint*, *i.e.*, $\alpha \cong \beta$, if and only if $|2h - \sum_{i=1}^7 E_i| \ne \emptyset$. We discuss the case $\alpha \cong \beta$ in the next section, and now assume that $\alpha \not\cong \beta$.

Proposition 2.2. *The multiplication map* $H^0(\alpha) \otimes H^0(\beta) \longrightarrow H^0(\alpha\beta) = H^0(K_C)$ *is surjective.*

Proof. Assume the contrary. Then there are two independent $(1,1)$-forms on $\mathbf{P}^2 \times \mathbf{P}^2$ vanishing on C. Let P be the pencil generated by them, and $X = X_P$ its base locus. If P contains a form of rank 1, then the image of $\Phi_{|\alpha|}$ is a line, which is a contradiction. Therefore P contains no $(1,1)$-forms of rank 1.

If P is regular, then X_P is irreducible (Proposition 4.1). Let $\pi : X_P \longrightarrow \mathbf{P}^2$ be the first projection. Then there is an effective divisor E such that $K_X = \pi^* K_{\mathbf{P}^2} + E$. On one hand, since $K_X = \mathcal{O}_X(-1,-1)$ and $\pi^* K_{\mathbf{P}^2} = \mathcal{O}_X(-3,0)$, we have $E.C = \deg \mathcal{O}_C(2,-1) = 7$. On the other hand, since I_P is of colength 3 (Proposition 4.2), we have a composition series $I_P = I_3 \subset I_2 \subset I_1 \subset \mathcal{O}_{\mathbf{P}^2}$ of ideal sheaves and a composition

$$\psi : X_3 \overset{\psi_3}{\to} X_2 \overset{\psi_2}{\to} X_1 \overset{\psi_1}{\to} \mathbf{P}^2$$

of three one-point-blow-ups. X_3 is smooth and its canonical class is $\psi^* K_{\mathbf{P}^2} + E_1 + E_2 + E_3$, where E_i is the total transform of the exceptional divisors of ψ_i. By the universal property of the blow-up ([H] II 7.14), there is a natural birational morphism $\phi : X_3 \longrightarrow X_P = Bl_{I_P} \mathbf{P}^2$. Since X_P is a complete linear section of $\mathbf{P}^2 \times \mathbf{P}^2$, it has at worst rational double points as its singularities. Thus ϕ is a crepant resolution, and we have $E_1 + E_2 + E_3 = \phi^* E$. Therefore, for some i, we have $E_i.C \geq 3$, and $\psi^* \mathcal{O}_{\mathbf{P}^2}(1) - E_i$ restricts to a g^1_d with $d \leq 4$ on C. This is a contradiction.

If P is singular, then X_P is either

$$\text{I)} \quad \Delta \cup (\mathbf{P}^1 \times \mathbf{P}^1), \quad \text{or} \quad \text{II)} \quad \mathbf{F}_{3,2} \cup (p \times \mathbf{P}^2)$$

by the table in §4. In the former case C is contained in either the diagonal Δ or $\mathbf{P}^1 \times \mathbf{P}^1$. This means that α is isomorphic to β or that the image of $\Phi_{|\alpha|}$ is a line. In the latter case we have either that the image of $\Phi_{|\beta|}$ is a conic or that the image of $\Phi_{|\alpha|}$ is a point. Thus we have a contradiction. □

Now, we consider the multiplication map

$$H^0(S,h) \otimes H^0\Big(S, 3h - \sum_{i=1}^{7} E_i\Big) \longrightarrow H^0\Big(S, 4h - \sum_{i=1}^{7} E_i\Big).$$

This is not injective since $h^0(3h - \sum_{i=1}^7 E_i) = h^0(\beta) = 3$ and $h^0(S, 4h - \sum_{i=1}^7 E_i) = h^0(K_C) = 8$ by Lemma 2.1. Similarly the dimension of the kernel of

$$H^0(S,2h) \otimes H^0\Big(S, 3h - \sum_{i=1}^{7} E_i\Big) \longrightarrow H^0\Big(S, 5h - \sum_{i=1}^{7} E_i\Big)$$

is at least $6 \times 3 - h^0(\alpha K_C) = 4$. Hence the image of the rational map

$$\big(\Phi_{|h|}, \Phi_{|3h - \sum E_i|}\big) : S \dashrightarrow \mathbf{P}^2 \times \mathbf{P}^2$$

is contained in a divisor W of bidegree $(1,1)$ and W' of bidegree $(2,1)$ such that $\dim W \cap W' = 2$. The pull-back of the divisor class of bidegree $(1,2)$ to S is $h + 2(3h - \sum_{i=1}^7 E_i) = 7h - 2\sum_{i=1}^7 E_i$ and linearly equivalent to C.

We now look at the 15 quadrics which vanish on the canonical model $C_{14} \subset \mathbf{P}^7$ of C. First, C_{14} is contained in a hyperplane section of the Segre variety

$$[W \subset \mathbf{P}^7] = [\mathbf{P}^2 \times \mathbf{P}^2 \subset \mathbf{P}^8] \cap H,$$

and there are 9 quadrics vanishing on W. Next, there are 3 quadrics which cut out $W \cap W'$ from W. Finally, since the pull-back of $\mathcal{O}_{\mathbf{P}^7}(2)$ to S is $\mathcal{O}_S\big(2(4h - \sum_{i=1}^7 E_i)\big) = \mathcal{O}_S(C + h)$, there are 3 more independent quadrics vanishing on C. Thus we have found $9 + 3 + 3 = 15$ independent quadrics vanishing on C. By Noether's theorem, they form a basis of $H^0(\mathbf{P}^7, \mathcal{I}_C(2))$, and by the Enriques-Petri theorem ([GH], Chap. 4), they define the canonical model $C_{14} \subset \mathbf{P}^7$ scheme-theoretically. Thus C is the complete intersection of divisors $(1,2)$ and $(2,1)$ in W (Case $\langle 6 \rangle$ of Theorem).

3. Curves with a self adjoint net. We assume that $\alpha^2 \simeq K_C$. Let $\Delta \subset S = S_{(7)}$ be the unique member of $\big|2h - \sum_{i=1}^7 E_i\big|$ and $\bar{\Delta} \subset \mathbf{P}^2$ its image. Then $\bar{\Delta}$ is an irreducible conic. We choose homogeneous coordinates of $\Delta \cong \bar{\Delta} \cong \mathbf{P}^1$ and $\mathbf{P}^2$ such that the morphism $\Delta \to \mathbf{P}^2$ is given by

$$(s:t) \mapsto (x_0 : x_1 : x_2) = (s^2 : st : t^2).$$

The surface S is the blow-up at seven points on $\bar{\Delta}$. Let $f(s,t) = 0$ be the equation of degree 7 over $\bar{\Delta}$ whose solutions are the seven points. We shall construct a polynomial $F(x) \in H^0\big(S, \mathcal{O}_S(7h - 2\sum_{i=1}^7 E_i)\big)$ which is determinantal in a certain sense. This will imply that the system of equations of $C \subset \mathbf{P}(11122)$ is 5×5 Pfaffian.

We start with a pair of ternary quartic polynomials $A(x)$ and $B(x)$ such that $A(s^2, st, t^2) = sf(s,t)$ and $B(s^2, st, t^2) = tf(s,t)$. Such polynomials exist by the exact sequence

(1)
$$H^0(\mathcal{O}_{\mathbf{P}}(2)) \to H^0(\mathcal{O}_{\mathbf{P}}(4)) \to H^0(\bar{\Delta}, \mathcal{O}_{\bar{\Delta}}(4)) \to 0$$
$$\cong$$
$$H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(8)).$$

Since $tA(s^2, st, t^2) - sB(s^2, st, t^2)$ is zero, the quintic polynomials $x_1A(x) - x_0B(x)$ and $x_2A(x) - x_1B(x)$ are divisible by $\delta(x)$, the equation of $\bar{\Delta} \subset \mathbf{P}^2$. We put

(2) $$\begin{cases} -x_0B(x) + x_1A(x) = \delta(x)D(x) \\ -x_1B(x) + x_2A(x) = \delta(x)E(x), \end{cases}$$

where $D(x)$ and $E(x)$ are cubic forms. Put

$$\begin{cases} D(x) = q_0(x)x_0 + q_1(x)x_1 + q_2(x)x_2 \\ E(x) = r_0(x)x_0 + r_1(x)x_1 + r_2(x)x_2 \end{cases}$$

for quadratic forms $q_i(x)$'s and $r_i(x)$'s. Then by Cramer's rule we have

(3)
$$\frac{\begin{vmatrix} -A + q_1\delta & q_2\delta \\ B + r_1\delta & -A + r_2\delta \end{vmatrix}}{x_0} = \frac{\begin{vmatrix} q_2\delta & B + q_0\delta \\ -A + r_2\delta & r_0\delta \end{vmatrix}}{x_1}$$
$$= \frac{\begin{vmatrix} B + q_0\delta & -A + q_1\delta \\ r_0\delta & B + r_1\delta \end{vmatrix}}{x_2} := F(x).$$

Here $F(x)$ is a form of degree 7 since $x_iF(x)$ is a form of degree 8 for $i = 0, 1, 2$. Let y_0, y_1 and z be new indeterminates which are algebraically independent over the field $k(x_0, x_1, x_2)$. We consider the ring homomorphism

$$\varphi_S : k[x_0, x_1, x_2, y_0, y_1, z] \longrightarrow k\left[x_0, x_1, x_2, \frac{1}{\delta(x)}\right],$$
$$y_0 \mapsto \frac{A(x)}{\delta(x)}, \quad y_1 \mapsto \frac{B(x)}{\delta(x)}, \quad z \mapsto \frac{F(x)}{\delta(x)^2}$$

and its kernel I_S. Then I_S is a (quasi-)homogeneous ideal under the grading $\deg x_i = 1$, $\deg y_j = 2$ and $\deg z = 3$. By the equation (2), two cubic forms

(4) $$\begin{matrix} a_0(x,y)x_0 + a_1(x,y)x_1 + a_2(x,y)x_2, \text{ and} \\ b_0(x,y)x_0 + b_1(x,y)x_1 + b_2(x,y)x_2 \end{matrix}$$

belong to I_S, where we put

$$a_0(x,y) = y_1 + q_0(x), \ a_1(x,y) = -y_0 + q_1(x), \cdots$$
$$\cdots, b_1(x,y) = y_1 + r_1(x), \ b_2(x,y) = -y_0 + r_2(x).$$

By (3), three quartic forms

(5)
$$x_0z - \begin{vmatrix} a_1(x,y) & a_2(x,y) \\ b_1(x,y) & b_2(x,y) \end{vmatrix}, \begin{vmatrix} a_2(x,y) & a_0(x,y) \\ b_2(x,y) & b_0(x,y) \end{vmatrix} - x_1z,$$
$$x_2z - \begin{vmatrix} a_0(x,y) & a_1(x,y) \\ b_0(x,y) & b_1(x,y) \end{vmatrix}$$

also belong to I_S. These five relations (4) and (5) are the five 4×4 Pfaffians of the skew-symmetric matrix

$$\begin{pmatrix} 0 & z & a_0(x,y) & a_1(x,y) & a_2(x,y) \\ & 0 & b_0(x,y) & b_1(x,y) & b_2(x,y) \\ & & 0 & x_2 & -x_1 \\ & \ominus & & 0 & x_0 \\ & & & & 0 \end{pmatrix}.$$

Now we relate the ideal I_S with the anti-canonical ring of a weak log del Pezzo surface. Let

$$R := \bigoplus_{n \geq 0} H^0\left(S, \left\lfloor n\left(h + \frac{2}{3}\Delta\right)\right\rfloor\right)$$

be the homogeneous coordinate ring of the $\mathbf{Q}$-divisor $h + (2/3)\Delta$, which is linearly equivalent to $-K_S - (1/3)\Delta$. For a global section $s \in H^0(S, n(h + (2/3)n)) = H^0(S, nh + a\Delta) = H^0((n + 2a)h - a\sum_{i=1}^7 E_i)$, $a = \lfloor (2/3)n \rfloor$, its push-forward $\pi_*s \in H^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(n + 2a))$ is a homogeneous polynomial of degree $n + 2a$. We identify R with the image of the injective ring homomorphism $\psi : R \longrightarrow k[x_0, x_1, x_2, 1/\delta(x)]$ defined by

$$H^0(S, nh + a\Delta) \ni s \mapsto \frac{\pi_*s}{\delta(x)^a} \in k\left[x_0, x_1, x_2, \frac{1}{\delta(x)}\right]_n.$$

The degree 1 part $H^0(S, h)$ is spanned by the homogeneous coordinates x_0, x_1, x_2. The degree 2 part $H^0(S, 2h + \Delta)$ contains $S^2\langle x_0, x_1, x_2\rangle$ as a subspace. The pull-back of the quartic forms $A(x)$ and $B(x)$ to S belong to $H^0(S, 4h - \sum_{i=1}^7 E_i)$ and $\{A(x)/\delta(x), B(x)/\delta(x)\}$ is a complementary basis of $S^2\langle x_0, x_1, x_2\rangle \subset H^0(S, \mathcal{O}_S(2h + \Delta))$ by the exact sequence (1).

Consider the multiplication map

(6) $$H^0(S, h) \otimes H^0(S, 2h + \Delta) \longrightarrow H^0(S, 3h + \Delta)$$

from degree 1 and 2 to degree 3. Since the restriction maps $H^0(S, h) \longrightarrow H^0(\mathcal{O}_\Delta(h))$ and $H^0(S, 2h + \Delta) \longrightarrow H^0(\mathcal{O}_\Delta(2h + \Delta))$ are surjective, so is this multiplication map. By the exact sequence

$$0 \to \mathcal{O}_S\left(5h - \sum_{i=1}^7 E_i\right) \to \mathcal{O}_S(3h + 2\Delta) \to \mathcal{O}_\Delta \to 0$$

and Lemma 2.1, the degree 3 part $H^0(S, 3h+2\Delta)$ is generated by the image of (6) and $F(x)/\delta(x)^2$.

Now we relate the ideal I_S with $C \in \left|7h - 2\sum_{i=1}^{7} E_i\right|$. Since C is disjoint from Δ and since $\mathcal{O}_C(h) \cong \alpha$, we have the restriction maps

$$(7) \qquad H^0\Big(S, \Big\lfloor n\Big(h+\frac{2}{3}\Delta\Big)\Big\rfloor\Big) \longrightarrow H^0(C, \alpha^n)$$

and $R \longrightarrow R(C,\alpha) := \bigoplus_{n\geq 0} H^0(C,\alpha^n)$. Since (7) is an isomorphism for $n=1$ and 2, the ring homomorphisms φ_S and ψ induce that

$$\varphi_C : k[x_0, x_1, x_2, y_0, y_1] \longrightarrow R(C,\alpha)$$

to the semi-canonical ring.

The equation of $\bar{C} \subset \mathbf{P}^2$, or $C \subset S$, is of the form $F(x) + \delta(x)G(x)$ for a quintic form $G(x) \in H^0(S, 5h - \sum_{i=1}^{7} E_i)$. There exist a cubic form $c(x, y_0, y_1)$ such that $c(x, A(x)/\delta(x), B(x)/\delta(x)) = G(x)/\delta(x)^2$ and a commutative diagram

$$\begin{array}{ccc} k[x_0,x_1,x_2,y_0,y_1,z] & \overset{\varphi_S}{\longrightarrow} & R \\ \downarrow & & \downarrow \\ k[x_0,x_1,x_2,y_0,y_1] & \underset{\varphi_C}{\longrightarrow} & R(C,\alpha), \end{array}$$

where the left vertical map is the specialization of z to the degree 3 element $c(x,y)$. Hence the five 4×4 Pfaffians of

$$(8) \qquad \begin{pmatrix} 0 & c(x,y) & a_0(x,y) & a_1(x,y) & a_2(x,y) \\ & 0 & b_0(x,y) & b_1(x,y) & b_2(x,y) \\ & & 0 & x_2 & -x_1 \\ & \ominus & & 0 & x_0 \\ & & & & 0 \end{pmatrix}$$

belongs to the kernel of φ_C.

Now we prove Theorem in Case $\langle 7 \rangle$. Let $C \subset \mathbf{P}^7 = \mathbf{P}^* H^0(K_C)$ be the canonical model of C. Since $\mathrm{Sym}^2 H^0(\alpha) \subset H^0(K_C)$, C is contained in the join of the Veronese surface and a line. This join is nothing but the weighted projective space $\mathbf{P}(11122)$ whose coordinates are x_0, x_1, x_2, y_0, y_1. Two polynomials (4) vanish on C. Multiplying these by x_0, x_1 and x_2, we obtain 6 relations of degree 4, which are linearly independent. Together with 3 relations (5) of degree 4, the five Pfaffians of (8) generate 9 quartic forms on $\mathbf{P}(11122)$ vanishing on C. On the other hand there are 6 quadratic forms vanishing on $\mathbf{P}(11122) \subset \mathbf{P}^7$. Hence we have 15 quadratic forms vanishing on $C \subset \mathbf{P}^7$. These are all quadratic forms vanishing on C. Hence the five Pfaffians cut out C scheme-theoretically from $\mathbf{P}(11122)$ by the Enriques-Petri theorem. Case $\langle 7 \rangle$ of Theorem follows since w-$G(2.5)$ is 5×5 Pfaffian in $\mathbf{P}(1^3 : 2^6 : 3)$.

4. Pencil of matrices. For a 3×3 matrix $A = (a_{ij})_{0\leq i,j\leq 2}$, we denote the divisor $f_A(x,y) := \sum_{0\leq i,j\leq 2} a_{ij}x_i y_j = 0$ in the Segre variety $\mathbf{P}^2 \times \mathbf{P}^2 \subset \mathbf{P}^8$ by X_A, where $(x_0 : x_1 : x_2)$ and $(y_0 : y_1 : y_2)$ are the homogeneous coordinates. Then X_A is reducible, singular at one point and smooth according as A is of rank 1, 2 and 3.

Let P be a 2-dimensional space of 3×3 matrices and $\{A, B\}$ be its basis. We classify $X_P := X_A \cap X_B$. We call P *regular* if it contains an invertible matrix and *singular* otherwise. Let

$$f_A(x,y) = a_0(x)y_0 + a_1(x)y_1 + a_2(x)y_2,$$
$$f_B(x,y) = b_0(x)y_0 + b_1(x)y_1 + b_2(x)y_2,$$

be the $(1,1)$-forms corresponding to A and B and I_P the ideal sheaf of $\mathcal{O}_{\mathbf{P}^2}$ generated by the minors

$$D_0 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}, \quad D_1 = \begin{vmatrix} a_2 & a_0 \\ b_2 & b_0 \end{vmatrix}, \quad D_2 = \begin{vmatrix} a_0 & a_1 \\ b_0 & b_1 \end{vmatrix},$$

of the coefficient matrix. Then the zero locus $V(I_P) \subset \mathbf{P}^2$ is the locus where the first projection $\pi : X_P \longrightarrow \mathbf{P}^2$ is not isomorphic.

If P is regular, then the divisor Y corresponding to an invertible matrix in P is nonsingular and the projections $Y \longrightarrow \mathbf{P}^2$ are $\mathbf{P}^1$-bundles. By the Lefshetz Theorem, the Picard number of Y is equal to 2 and the Picard group is generated by $\mathcal{O}_Y(1,0)$ and $\mathcal{O}_Y(0,1)$. Thus if X_P is reducible, it must be a sum of divisors of bidegree $(1,0)$ and $(0,1)$ on Y, *i.e.*, a section of Y by $(1,1)$-forms of rank 1. X_P is the union of two cubic scrolls $\mathbf{F}_{2,1} \cup \mathbf{F}_{1,2}$. So we have

Proposition 4.1. *If P is regular and contains no member of rank 1, then X_P is irreducible.*

It is well known that π is the blow-up at three points if X_P is smooth.

Proposition 4.2. *Let P and X_P be as in the above proposition. Then $V(I_P)$ is of dimension 0, and $I_P \subset \mathcal{O}_{\mathbf{P}^2}$ is of colength 3. Moreover $\pi : X_P \longrightarrow \mathbf{P}^2$ is the blowing-up with center I_P.*

Proof. If $\dim V(I_P) > 0$, then the inverse image $\pi^{-1}V(I_P)$ is a surface. This is impossible since X_P is irreducible of dimension 2. The colength of I_P is equal to 3 since it is so if X_P is smooth.

The blow-up $Bl_{I_P}\mathbf{P}^2$ with center $V(I_P)$ has a natural embedding φ into $\mathbf{P}^2 \times \mathbf{P}^2$. φ is an isomorphism onto X_P since $a_0D_0 + a_1D_1 + a_2D_2 = b_0D_0 +$

$b_1D_1 + b_2D_2 = 0$ and since X_P is irreducible and reduced. $\square$

If P is singular, then by Kronecker's classification ([Ga] Chap. XII), P is either

$$\text{type I} \quad \left\langle \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \right\rangle,$$

$$\text{type II} \quad \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \right\rangle, \text{ or}$$

$$\text{type III} \quad \left\langle \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\rangle$$

modulo suitable linear transformations and modulo transpose.

If P is of type I, then the defining equation of X_P is

$$x_0y_1 - x_1y_0 = x_0y_2 - x_2y_0 = 0,$$

and X_P is the union of the diagonal Δ and $\mathbf{P}^1 \times \mathbf{P}^1$. If P is of type II, then the equation is

$$x_0y_0 - x_1y_1 = x_0y_0 - x_1y_2 = 0,$$

and X_P is the union of

$$\overline{\{(1 : \lambda : \mu) \times (\lambda^2 : \lambda : 1) \mid \lambda, \mu \in k\}},$$

a quintic scroll $\mathbf{F}_{3,2}$, and a plane $p \times \mathbf{P}^2$. All non-zero members are of rank 2 in these cases.

If P is of type III, then by the Jordan normal form of 2×2-matrices, the defining equation of X_P is either

$$x_0y_0 + x_1y_1 = x_iy_j = 0 \quad \text{for some } 0 \le i \le j \le 1, \text{ or}$$
$$x_0y_0 = x_0y_1 = 0.$$

So we have the following table:

P	rank 1	X_P	degree
reg.	$\nexists$	$Bl_{I_P}\mathbf{P}^2$	6
	$\exists$	$\mathbf{F}_{2,1} \cup \mathbf{F}_{1,2}$	3+3
sing.	$\nexists$	$\Delta \cup (\mathbf{P}^1 \times \mathbf{P}^1)$	4+2
		$\mathbf{F}_{3,2} \cup (p \times \mathbf{P}^2)$	5+1
	$\exists$	$2\mathbf{P}^2 \cup 2(\mathbf{P}^1 \times \mathbf{P}^1)$	1+1+2+2
		$(\mathbf{P}^1 \times \mathbf{P}^2) \cup (\mathbf{P}^2 \times p)$	(3)+1

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