The existence of plane curves with prescribed singularities

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Abstract: By improving the vanishing theorem of Hirschowitz, we prove an existence theorem of the reduced irreducible plane curve with ordinary singularities at given points in general position, which improves an earlier result by Greuel, Lossen and Shustin.

Key words: Plane curve; singularity; linear system; base condition.

1. Introduction. Let $d, m_1, \dots, m_r$ $(r \geq 1)$ be non-negative integers and $P_1, \dots, P_r \in \mathbf{P}^2$. Assuming that the r points are in general position on $\mathbf{P}^2$, we deal with the problem: when does there exist a reduced and irreducible plane curve of degree d with an ordinary singular point of multiplicity m_i at P_i for all i? A result of Greuel, Lossen and Shustin [2] guarantees that such a curve exists if

$$\sum_{i=1}^{r} \frac{1}{2} m_i(m_i+1) < \frac{1}{4}d^2 + \frac{3}{2}d - \frac{1}{4} - \left[\frac{d}{2}\right],$$

where $[x]$ denotes the integer such that $[x] \leq x < [x]+1$. On the other hand, a simple dimension count shows that there exists a curve of degree d with multiplicity $\geq m_i$ at P_i for all i if $\sum_{i=1}^{r} m_i(m_i+1)/2 \leq d(d+3)/2$. We are interested in making the right hand side of the inequality "larger" while keeping the existence theorem of the former type.

Our Corollary 11 gives the existence theorem with

$$\left[\frac{1}{3}(d-1)^2\right] + 2d - 1$$

on the right hand side with certain obvious necessary conditions. In order to prove this result, we refine the vanishing theorem by Hirschowitz [3] as Corollary 7.

We consider all the objects in this paper to be defined over an algebraically closed field K. $\mathbf{Z}_{\geq 0}$ and $\mathbf{Z}_{>0}$ denote the sets of non-negative and positive integers, respectively.

2. Vanishing theorem. Let $f : S \to \mathbf{P}^2$ be the blowing up of the projective plane at r distinct points $P_1, \dots, P_r$ and let $H = f^*\mathcal{O}_{\mathbf{P}^2}(1)$, and E_i the exceptional divisor $f^{-1}(P_i)$ for all i. Let L_i be a generic line through P_i on $\mathbf{P}^2$ and $L_i' \subset S$ the strict transform of L_i for all i. Let L_{ij} be the line through P_i and P_j, and L_{ij}' the strict transform of L_{ij}. Let $\mathcal{O}(d, m) = \mathcal{O}_S(dH - \sum_{i=1}^{r} m_i E_i)$. Note that we use the vector notation $m = (m_1, \dots, m_r)$, if there is no danger of confusion. We also use the notation $m_J = \sum_{j \in J} m_j$ for subset $J \subset \mathbf{Z}_{>0}$ if the right hand side makes sense.

The vanishing theorem is the following:

Theorem 1. *Let L be a line on $\mathbf{P}^2$. Let $I = I(L) = \{i \in [1, r] \mid P_i \in L\}$. Assume that the points $\{P_i \mid i \notin I\}$ are in general position. Let*

$$M(d) = \left[\frac{1}{3}d^2\right] + 2d + 1.$$

If $d, m_1, \dots, m_r \in \mathbf{Z}_{\geq 0}$ satisfy the following conditions (a)–(d), *then $H^1(S, \mathcal{O}(d, m)) = 0$.*

(a) $\sum_{i=1}^{r} m_i(m_i+1)/2 \leq M(d)$,
(b) $m_J \leq d+1$ *for all $J \subset [1, r]$ with $|J| = 2$,*
(c) $m_I \leq d+1$,
(d) $m_J \leq 2d+1$ *for all $J \supset I$ with $|J| = |I|+2$.*

Before proving this theorem, we prepare some easy lemmas.

Lemma 2. *Let $d, m_1, \dots, m_r \in \mathbf{Z}_{\geq 0}$ satisfy the inequality: $\sum_{i=1}^{r} m_i(m_i+1)/2 \leq (d+1)(d+2)/2$.*

(1) *If $r \geq 4$ and there exist j, k $(j \neq k)$ with $m_j + m_k = d+1$ then $m_p + m_q \leq d$ for all distinct $p, q \notin \{j, k\}$.*
(2) *If $r \geq 5$ and there exist distinct indices j, k, l such that $m_j + m_k = d+1$, $m_l > 0$, then $m_p + m_q \leq d-1$ for all distinct $p, q \notin \{j, k, l\}$.*

Proof. These assertions are obvious in view of the following inequalities in $a, b, d \in \mathbf{Z}_{\geq 0}$:

$$\begin{cases} (a(a+1) + b(b+1))/2 \geq [(a+b+1)^2/4], \\ (d+1)(d+2)/2 = [(d+2)^2/4] + [(d+1)^2/4]. \end{cases}$$

$\square$

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Lemma 3. *Assume that $d, m_1, \dots, m_r \in \mathbf{Z}_{\geq 0}$ satisfy $\sum_{i=1}^{r} m_i(m_i+1)/2 \leq [(d+2)^2/3] - 1$. If $r \geq 3$ and there exist distinct j, k such that $m_j + m_k \geq d+1$ then $m_j + m_p, m_k + m_p \leq d$ for all $p \notin \{j, k\}$.*

Proof. We may assume $m_j \geq m_k$. Hence it is sufficient to show that $m_j + m_p \leq d$ $(\forall p \notin \{j,k\})$.

If there exists $p \notin \{j,k\}$ such that $m_j + m_p \geq d+1$, we can derive a contradiction as in the following calculation using $m_k, m_p \geq d+1-m_j$:

$$\begin{aligned}\sum_{i=1}^{r} \frac{1}{2} m_i(m_i+1) &\geq \sum_{i \in \{j,k,p\}} \frac{1}{2} m_i(m_i+1) \\ &\geq \frac{3}{2}\left(m_j - \frac{2}{3}d - \frac{5}{6}\right)^2 + \frac{1}{3}(d+2)^2 - \frac{3}{8} \\ &\geq \left[\frac{1}{3}(d+2)^2\right].\end{aligned}$$

□

Next two lemmas are easy vanishing lemmas.

Lemma 4. *Let T be a nonsingular surface. Let D, E be divisors on T with $(D+E)E \geq -1$ and $E \simeq \mathbf{P}^1$. If $H^1(T, \mathcal{O}_T(D)) = 0$ then $H^1(T, \mathcal{O}_T(D+E)) = 0$.*

Lemma 5. *Let S, d, m, $\mathcal{O}(d,m)$ be as in Theorem 1. Assume that $d, m_1, \dots, m_r$ satisfy the following (1) or (2), then $H^1(S, \mathcal{O}(d,m)) = 0$ and $|\mathcal{O}(d+1,m)|$ is base point free.*

(1) $m_{[1,r]} \leq d+1$.

(2) *$r \geq 2$, $m_{[2,r]} \leq d+1$, $m_{\{1,i\}} \leq d+1$, and $L'_{1i} \sim H - E_1 - E_i$ for all $i > 1$ (in other words, L_{1i} passes through only P_1 and P_i).*

Proof. If $m_i = 0$ for all i, then

$$H^1(S, \mathcal{O}(d,m)) = H^1(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(d)) = 0,$$

and $|\mathcal{O}(d+1,m)| = |\mathcal{O}_{\mathbf{P}^2}(d+1)|$ is base point free. We may thus assume $m_i > 0$ for some i below.

We prove the lemma by induction on d. If $d = 0$, then $\mathcal{O}(d,m) = \mathcal{O}_S(-E_i)$ for some i by the hypotheses. Hence obviously $H^1(S, \mathcal{O}(d,m)) = 0$ and $|\mathcal{O}(d+1,m)|$ is base point free by $\mathcal{O}(d+1,m) = \mathcal{O}_S(H - E_i)$.

Assume $d \geq 1$. When $d, m_1, \dots, m_r$ satisfy (1), we may assume $m_1 > 0$ by renumbering m_i's. Let $m'_1 = m_1 - 1$, $m'_i = m_i$ $(2 \leq i \leq r)$. By $L'_1 \sim H - E_1$, we have

$$\mathcal{O}(d,m) \simeq \mathcal{O}(d-1,m')(L'_1).$$

Since $\sum_{i=1}^{r} m'_i = \sum_{i=1}^{r} m_i - 1 \leq d$, $H^1(S, \mathcal{O}(d-1,m')) = 0$ holds by the induction assumption. Hence $H^1(S, \mathcal{O}(d,m)) = 0$ by Lemma 4 because $(dH - \sum_{i=1}^{r} m_i E_i)L'_1 = d - m_1 \geq -1$ and $L'_1 \simeq \mathbf{P}^1$. Since $|\mathcal{O}(d,m')|$ is base point free by the induction assumption and $|L'_1| = |H - E_1|$ is base point free, we see that $|\mathcal{O}(d+1,m)| = |\mathcal{O}(d,m')(L'_1)|$ is base point free.

When d, m and $P = (P_1, \dots, P_r)$ satisfy (2), we may assume $m_{[1,r]} \geq d+2$ by (1). Hence $m_1 = m_{[1,r]} - m_{[2,r]} > 0$. Similarly there exist distinct $j, k \geq 2$ with $m_j, m_k > 0$ because $\sum_{i \neq 1,a} m_i = m_{[1,r]} - m_{\{1,a\}} > 0$ for each $a > 1$. Hence we may assume $m_2 > 0$ and $m_3 > 0$ by renumbering m_i's. Let $m'_i = m_i - 1$ $(i = 1,2)$, $m'_i = m_i$ $(3 \leq i \leq r)$. Since $L'_{12} \sim H - E_1 - E_2$, we have

$$\mathcal{O}(d,m) \simeq \mathcal{O}(d-1,m')(L'_{12}).$$

Since $m'_{[2,r]} = m_{[2,r]} - 1 \leq d$ and $m'_{\{1,i\}} \leq m_{\{1,i\}} - 1 \leq d$ $(2 \leq \forall i \leq r)$, we have $H^1(S, \mathcal{O}(d-1,m')) = 0$ by the induction assumption. By Lemma 4, we have $H^1(S, \mathcal{O}(d,m)) = 0$ because $(dH - \sum_{i=1}^{r} m_i E_i)L'_{12} = d - m_1 - m_2 \geq -1$ and $L'_{12} \simeq \mathbf{P}^1$. Similarly, let $m''_i = m_i - 1$ $(i = 1,3)$ and $m''_i = m_i$ $(i \neq 1,3)$. Since $L'_{13} \sim H - E_1 - E_3$, we have

$$\mathcal{O}(d+1,m) \simeq \mathcal{O}(d,m'')(L'_{13}).$$

Because $|\mathcal{O}(d,m')|$ and $|\mathcal{O}(d,m'')|$ are base point free by the induction assumption and $L'_{12} \cap L'_{13} = \emptyset$, we see that $|\mathcal{O}(d+1,m)|$ is base point free. □

We begin with an extreme case of Theorem 1.

Lemma 6. *Let S, d, m, $\mathcal{O}(d,m)$ be as in Theorem 1. Assume that $m_i \geq d+1$ for some i, say $i = 1$. Then $d \leq 3$, $m_1 = d+1$, $m_i = 0$ for all $i > 1$. In this case, $H^1(S, \mathcal{O}(d,m)) = 0$ holds.*

Proof. By the condition (a) and $m_1 \geq d+1$, we have

$$\frac{1}{2}(d+1)(d+2) + m_{[2,r]} \leq \frac{1}{3}d^2 + 2d + 1,$$

which reduces to $m_{[2,r]} \leq d(3-d)/6$. Hence $d \leq 3$, $m_{[2,r]} = 0$ and $m_1 = d+1$. The last assertion follows from $m_{[1,r]} = d+1$ and Lemma 5, (1). □

Let us prove Theorem 1.

Proof. First we may assume $d \geq m_i > 0$ for all i. Indeed if $m_i = 0$ for some i, then deleting P_i weakens the hypothesis of the theorem and has no effect on $H^1(S, \mathcal{O}(d,m)) = 0$. Thus we may assume $m_i > 0$. The other inequality $d \geq m_i$ follows from Lemma 6.

We may assume $r \geq 4$ since this theorem is proved by Lemma 5 and the condition (b) when $r \leq 3$.

Let $t_d = M(d) - M(d-1)$ $(d \geq 1)$. It is easy to see that $t_d = [2d/3] + 2$ and that $t_d \leq d$ (if $d \geq 4$).

We prove this theorem by induction on d. If $d = 0$, $H^1(S, \mathcal{O}(d, m)) = 0$ holds by Lemma 5 (1) since $M(0) = 1$.

Assume $d \geq 1$. First, we consider the restriction to L.

Case A: There exists a subset $I'(\supset I)$ of $[1, r]$ such that $t_d \leq m_{I'} \leq d+1$ and $m_{\{k,l\}} \leq d$ for all distinct $k, l \notin I'$.

If this theorem holds for the more special situation $I(L) = I' \supset I$, then H^1 also vanish for the more general situation $I(L) = I$ by the upper semicontinuity of h^1. Thus we may assume the conditions: $t_d \leq m_I$ and $m_{\{k,l\}} \leq d$ for all distinct $k, l \notin I$.

We define $m'_i \in \mathbf{Z}_{\geq 0}$ $(1 \leq i \leq r)$ by

$$m'_i = \begin{cases} m_i - 1 & (i \in I), \\ m_i & (i \notin I). \end{cases}$$

Since $m'_I = m_I - |I|$ and $m'_{\{k,l\}} = m_{\{k,l\}} \leq d$ for all distinct $k, l \notin I$ (the assumption of Case A), we have the following two inequalities:

$$(*) \quad \begin{cases} m'_I & \leq d-1, \\ m'_{\{k,l\}} \leq & d \quad (\forall k, l \notin I \text{ with } k \neq l), \end{cases}$$

Let $L' \subset S$ be the strict transform of L. Since $L' \sim H - \sum_{i \in I} E_i$, we can represent

$$\mathcal{O}(d, m) \simeq \mathcal{O}(d-1, m')(L').$$

Moreover $(dH - \sum_{i=1}^r m_i E_i)L' = d - m_I \geq -1$ and $L' \simeq \mathbf{P}^1$. Hence if $H^1(S, \mathcal{O}(d-1, m')) = 0$ holds then $H^1(S, \mathcal{O}(d, m)) = 0$ holds by Lemma 4. We will show that $d-1$ and m' satisfy the conditions (a)–(d). Indeed the condition (a) follows from

$$\sum_{i=1}^r \frac{1}{2} m'_i(m'_i + 1) = \sum_{i=1}^r \frac{1}{2} m_i(m_i+1) - \sum_{i \in I} m_i$$
$$\leq M(d) - t_d = M(d-1),$$

(b) from

$$m'_{\{i,j\}} \begin{cases} = m_{\{i,j\}} \leq d & \text{if } i, j \notin I, \\ \leq m_{\{i,j\}} - 1 \leq d & \text{if } i \text{ or } j \in I, \end{cases}$$

where $i \neq j$, and (c) and (d) are obvious by $(*)$.

Next, we consider the restriction to L_{ij}.

Case B: $m_I \leq d-1$ and there exist distinct $p, q \in [1, r]$ such that $t_d \leq m_{\{p,q\}}$.

We may assume $m_{\{i,j\}} \leq m_{\{p,q\}}$ for all distinct i, j. We define $m''_i \in \mathbf{Z}_{\geq 0}$ $(1 \leq i \leq r)$ by

$$m''_i = \begin{cases} m_i - 1 & (i = p \text{ or } q), \\ m_i & (\text{otherwise}). \end{cases}$$

We then claim the inequalities:

$$(*') \quad \begin{cases} m''_I & \leq d-1, \\ m''_{\{i,j\}} \leq & d \quad (1 \leq \forall i < \forall j \leq r). \end{cases}$$

First, $m''_I \leq m_I \leq d-1$ by the assumption of Case B. If $m_{\{p,q\}} \leq d$, then $m''_{\{i,j\}} \leq m_{\{i,j\}} \leq m_{\{p,q\}} \leq d$. Thus we assume $m_{\{p,q\}} = d+1$. If $\{i,j\} \cap \{p,q\} = \emptyset$, then $m''_{\{i,j\}} = m_{\{i,j\}} \leq d$ by Lemma 2, (1). If $\{i,j\} \cap \{p,q\} \neq \emptyset$, then $m''_{\{i,j\}} \leq m_{\{i,j\}} - 1 \leq m_{\{p,q\}} - 1 = d$ by the definition of m''. This proves our claim $(*')$.

If $p, q \in I$, $t_d \leq m_{\{p,q\}} \leq m_I \leq d+1$ and $m_{\{k,l\}} \leq d$ ($\forall k, l \notin \{p,q\}$ with $k \neq l$). Hence if $p, q \in I$ we are done by Case A, and it is enough to assume $p \notin I$ or $q \notin I$. Then P_p or P_q is in general position outside L and $L'_{pq} \sim H - E_p - E_q$.

Now, we can express $\mathcal{O}(d, m)$ as

$$\mathcal{O}(d, m) \simeq \mathcal{O}(d-1, m'')(L'_{pq}).$$

And $(dH - \sum_{i=1}^r m_i E_i)L'_{pq} = d - m_p - m_q \geq -1$ and $L'_{pq} \simeq \mathbf{P}^1$, hence, like in Case A it is sufficient to show that $d-1$ and m'' satisfy the conditions (a)–(d). The condition (a) holds

$$\sum_{i=1}^r \frac{1}{2} m''_i(m''_i + 1) = \sum_{i=1}^r \frac{1}{2} m_i(m_i+1) - m_p - m_q$$
$$\leq M(d) - t_d = M(d-1),$$

and (b)–(d) are obvious by $(*')$.

Case C: $m_I = d$ and there exist distinct $p, q \notin I$ such that $m_{\{p,q\}} = d+1$.

Note that P_p and P_q are in general position outside L and $L'_{pq} \sim H - E_p - E_q$ since $p, q \notin I$.

As in Case B, we define $m''_i \in \mathbf{Z}_{\geq 0}$ $(1 \leq i \leq r)$ by

$$m''_i = \begin{cases} m_i - 1 & (i = p \text{ or } q), \\ m_i & (\text{otherwise}). \end{cases}$$

We then claim the following:

$$(\star) \quad \begin{cases} m''_I & = \quad d, \\ m''_{\{i,j\}} \leq & \quad d \quad (1 \leq \forall i < \forall j \leq r), \\ m''_{\{k,l\}} \leq & d-1 \quad (\forall k, l \notin I \text{ with } k \neq l). \end{cases}$$

First, $m''_I = m_I = d$ and we can see $m''_{\{i,j\}} \leq d$ if $i \neq j$ as in the proof of $(*')$. It remains to prove the last inequality. Let distinct $k, l \notin I$. If $|\{k,l\} \cap \{p,q\}| = 0$, then $m''_{\{k,l\}} = m_{\{k,l\}} \leq d-1$ by Lemma 2, (2), since there exist at least 5 distinct indices k, l, p, q

and an element $i \in I$. If $|\{k,l\} \cap \{p,q\}| = 1$ (say, $l = p$), then $m''_{\{k,l\}} = m_{\{k,l\}} - 1 \le d-1$ follows from Lemma 3 since $k, p, q \notin I$ and

$$\begin{aligned}
&\sum_{i\in\{k,p,q\}} \frac{1}{2} m_i(m_i+1) \\
&\le \sum_{i=1}^{r} \frac{1}{2} m_i(m_i+1) - \sum_{i\in I} \frac{1}{2} m_i(m_i+1) \\
&\le M(d) - m_I = M(d) - d \le \left[\frac{1}{3}(d+2)^2\right] - 1.
\end{aligned}$$

Finally if $|\{k,l\} \cap \{p,q\}| = 2$, then $m''_{\{k,l\}} = m_{\{k,l\}} - 2 = d-1$. Thus our claim ($\star$) is proved.

We show that $d-1$ and m'' satisfy the conditions of the theorem. The condition (a) holds since $m_{\{p,q\}} = d+1 \ge t_d$ and (b)–(d) is clear by ($\star$).

Finally, we consider the case, in which we apply Lemma 5.

<u>*Case* D</u>: otherwise.

When $m_{[1,r]} \le d+1$, the theorem follows from Lemma 5. We may assume $m_{[1,r]} \ge d+2$. We note that $t_d \le d$ if $d \ge 4$ and that $t_d = d+1$ if $d = 1, 2, 3$.

Denying *Case* A, we have the following (A$_1$), (A$_2$) or (A$_3$) by the condition (c).

$$\begin{aligned}
&(\mathrm{A}_1) \quad m_{\{k,l\}} = d+1 \ (\exists k, l \notin I \text{ with } k \ne l),\\
&(\mathrm{A}_2) \quad m_I \le t_d - 1, \quad m_I \le d-1,\\
&(\mathrm{A}_3) \quad m_I = t_d - 1 = d, \quad 1 \le d \le 3.
\end{aligned}$$

We will treat these three cases separately. In Case (A$_1$), we deny *Case* B and obtain

$$(\mathrm{AB}_1) \quad \begin{cases} m_{\{k,l\}} = d+1 \ (\exists k, l \notin I \text{ with } k \ne l), \\ m_I = d, \end{cases}$$

using the condition (d). The case (A$_1$) is done, since (AB$_1$) is covered by *Case* C.

In Case (A$_2$), we also deny *Case* B and obtain

$$(\mathrm{AB}_2) \quad \begin{cases} m_I \le t_d - 1, \quad m_I \le d-1, \\ m_{\{i,j\}} \le t_d - 1 \quad (1 \le \forall i < \forall j \le r). \end{cases}$$

Since we denied *Case* A, we have $m_{I'} \le t_d - 1$ or $m_{I'} \ge d+2$ or $m_{\{k,l\}} = d+1$ ($\exists k, l \notin I'$ with $k \ne l$) for all subset $I' (\supset I)$ of $[1,r]$. Combining this with (AB$_2$), we have

$$m_{I'} \le t_d - 1 \ \text{ or } \ m_{I'} \ge d+2 \ \ ([1,r] \supset \forall I' \supset I).$$

Let J be a maximal element of

$$\{I' \subset [1,r] \mid m_{I'} \le t_d - 1,\ I \subset I'\}.$$

Since $m_I \le t_d - 1$ and $m_{[1,r]} \ge d+2$, we claim J has the following properties:

$$(**) \quad \begin{cases} m_J \le t_d - 1, \\ m_J + m_j \ge d+2 \quad (\forall j \notin J), \\ |J| = r-1. \end{cases}$$

The first two of ($**$) are obvious since J is maximal. Hence, m_j ($\forall j \notin J$) has the following lower bound:

$$\begin{aligned}
m_j = (m_J + m_j) - m_J &\ge d + 2 - (t_d - 1) \\
&= d + 3 - t_d.
\end{aligned}$$

If there are two distinct $j_1, j_2 \in [1,r] \setminus J$, then

$$t_d - 1 \ge m_{\{j_1,j_2\}} \ge 2(d+3-t_d),$$

which contradicts $t_d = [2d/3] + 2$. Hence, $|J| = r-1$ as claimed. Renumbering m_i's, we may assume $J = [2,r]$. Since $P_1 \notin L$ is in general position, the conditions of Lemma 5, (2) are satisfied and we have $H^1(S, \mathcal{O}(d,m)) = 0$ by the lemma. Thus Case (A$_2$) is done.

In Case (A$_3$), we deny *Case* C and obtain

$$(\mathrm{AC}_3) \quad \begin{cases} m_I = t_d - 1 = d, \quad 1 \le d \le 3, \\ m_{\{p,q\}} \le d \ \ (\forall p, q \notin I \ \text{ with } \ p \ne q). \end{cases}$$

If $|I| \le r-2$, then $m_p = 1$ for some $p \notin I$ since $m_p + m_q \le d \le 3$. Then with $I' = I \cup \{p\}$, we are in *Case* A and we are done. If $|I| \ge r-1$, then we are done by Lemma 5. Thus Case (A$_3$) is done. □

When all the points are in general position, the vanishing theorem needs fewer conditions.

Corollary 7. *Let d, m, $M(d)$ be as in Theorem 1. Let $P_1, \ldots, P_r \in \mathbf{P}^2$ be r distinct points in general position. If d, m satisfy the following conditions* (a) *and* (b), *then $H^1(S, \mathcal{O}(d,m)) = 0$.*

(a) $\sum_{i=1}^{r} m_i(m_i+1)/2 \le M(d)$,

(b) $m_{\{i,j\}} \le d+1$ $(1 \le \forall i < \forall j \le r)$.

Proof. Let L be a line not containing any of P_i's on $\mathbf{P}^2$. Then $I = \emptyset$ and the conditions of Theorem 1 are satisfied. □

Adding the upper bound of m: $m_i \le m_0$ $(1 \le \forall i \le r)$, we can improve $M(d)$ in Theorem 1. The following Corollary is Ballico's result [1].

Corollary 8. *Let d, m, P, L, I be as in Theorem 1. Let $m_i \le m_0$ $(1 \le \forall i \le r)$. Let*

$$M'(d) = \frac{1}{2}(d+1)(d+2) - (m_0 - 1)d.$$

If d, m satisfy the following conditions (a) *and* (b), *then $H^1(S, \mathcal{O}(d,m)) = 0$.*

(a) $\sum_{i=1}^{r} m_i(m_i+1)/2 \le M'(d)$,

(b) $m_I \le d+1$.

Proof. We use induction on d. When $d = 0$ it is clear. Assume $d \ge 1$. We may assume $m_{[1,r]} \ge d+$

2 by Lemma 5, (1). Since $m_I \le d+1$, there exists a subset $I'(\supset I)$ of $[1,r]$ such that $m_{I'} \le d+1$ and $m_{I'} + m_j \ge d+2$ for every $j \in [1,r] \setminus I'$. Hence

$$d+2-m_0 \le m_{I'} \le d+1.$$

We can use the argument used in *Case* A of Theorem 1 since $M'(d)$ has the following property:

$$M'(d) = M'(d-1) + (d+2-m_0) \quad (d \ge 1).$$

□

3. Existence theorem. In this section, K is assumed to be of characteristic zero. Let $d, m_1, \dots, m_r$ be positive integers.

$S_d(P,m)$ denotes the set of reduced irreducible curves of degree d with an ordinary singular point of multiplicity m_i at P_i for each i as their only singularities.

Theorem 9. *Let $L, I = I(L)$, $P_1, \dots, P_r, M(d)$ be as in Theorem* 1. *Assume that* $\mathrm{char}(K) = 0$ *and that $d, m_1, \dots, m_r \in \mathbf{Z}_{>0}$ satisfy the following conditions* (a)–(e). *Then there exists a curve $C \in S_d(P,m)$ transversal to L and L_{ij} for all i, j.*

(a) $\sum_{i=1}^{r} m_i(m_i+1)/2 \le M(d-1)$,
(b) $m_J \le d$ *for all $J \subset [1,r]$ with $|J| = 2$,*
(c) $m_I \le d$,
(d) $m_J \le 2d-1$ *for all $J \supset I$ with $|J| = |I|+2$.*
(e) $m \ne (d)$ (*i.e.* $r > 1$ *or* $m_1 \ne d$).

Remark 10. If $d, m_1, \dots, m_r \in \mathbf{Z}_{>0}$ satisfy the conditions (a) and (e) above, note that we have

$$(\mathcal{O}(d,m)^2) = d^2 - \sum_i m_i^2 > 0.$$

Indeed if $m_{[1,r]} \le d$, the inequality easily follows from (e). If $m_{[1,r]} \ge d+1$, it follows from (a) and

$$d^2 - \sum_i m_i^2 > 2M(d-1) - \sum_i m_i(m_i+1).$$

Proof of Theorem 9. When $1 \le r \le 3$, $|\mathcal{O}(d,m)|$ is base point free by Lemma 5 and the hypothesis (b). Hence a general member $C \in |\mathcal{O}(d,m)|$ is smooth and transversal to L' and L'_{ij} for all i, j by Bertini's Theorem and is irreducible since $C^2 = d^2 - \sum_{i=1}^{r} m_i^2 > 0$. Hence $f(C) \in S_d(P,m)$ has the required properties. We may assume $r \ge 4$. We prove this theorem by induction on d. We assume $d \ge 4$ for simplicity.

Our division and arguments will be similar to those in the proof of Theorem 1.

Case A: There exists a subset $I'(\supset I)$ of $[1,r]$ such that $t_{d-1} \le m_{I'} \le d$ and $m_{\{k,l\}} \le d-1$ for all distinct $k, l \notin I'$.

First of all, if this theorem holds for L with $I(L) = I'$, then the original theorem holds by the argument in the proof of [2, Lemma 3.3.1] since $H^1(S, \mathcal{O}(d,m)) = 0$ by Theorem 1 and the upper semi-continuity of h^1. Hence we may assume: $t_{d-1} \le m_I$ and $m_{\{k,l\}} \le d-1$ for all distinct $k, l \notin I$.

We define $m'_i \in \mathbf{Z}_{\ge 0}$ $(1 \le i \le r)$ by

$$m'_i = \begin{cases} m_i - 1 & (i \in I), \\ m_i & (i \notin I). \end{cases}$$

We omit the index i with $m'_i = 0$ and renumber the remaining m'_i's. Assume that there exists j such that $m'_j = d-1$. Then we see $I = [1,r] \setminus \{j\}$. Since $m_{[1,r]\setminus\{j\}} \le d$ and (b), we see that $|\mathcal{O}(d,m)|$ is free by Lemma 5, (2). Hence $f(C) \in S_d(d,m)$ is the required curve if C is a general member of $|\mathcal{O}(d,m)|$.

Now we assume $m' \ne (d-1)$. Using the argument used in *Case* A of Theorem 1, we see that $d-1$, m' satisfy the conditions (a)–(e) of the theorem. Hence, there exists a curve $C_{d-1} \in S_{d-1}(P,m')$ which is transversal to L and L_{ij} for all i, j by the induction assumption.

Let $s = d+1-m_I$ and $P_{r+1}, \dots, P_{r+s}$ be s distinct points on L outside C_{d-1} and $m_{r+1} = \cdots = m_{r+s} = 1$. Let $\overline{I} = I \cup \{r+1, \dots, r+s\}$. We check that $d, m_1, \dots, m_{r+s}$, $\{P_1, \dots, P_{r+s}\}, \overline{I}$ satisfy the conditions of Theorem 1. The condition (a) is satisfied:

$$\begin{aligned}\sum_{i=1}^{r+s} \frac{1}{2} m_i(m_i+1) &= \sum_{i=1}^{r} \frac{1}{2} m_i(m_i+1) + s \\ &\le M(d-1) + d + 1 - m_I \\ &\le M(d-1) + d + 1 - t_{d-1} \\ &\le M(d-1) + t_d = M(d).\end{aligned}$$

If $1 \le i < j \le r$, then $m_{\{i,j\}} \le d$ by the hypothesis (b). If $1 \le i \le r$ and $r+1 \le j \le r+s$, then $m_{\{i,j\}} = m_i + 1 \le d$ since $m_i \le d-1$ by the hypotheses (b) and (e). If $r+1 \le i < j \le r+s$, then $m_{\{i,j\}} = 2 \le d$. Hence $d, m_1, \dots, m_{r+s}$ satisfy the stronger condition (b$'$) $m_{\{i,j\}} \le d$ $(1 \le \forall i < \forall j \le r+s)$. The condition (c) is satisfied because

$$m_{\overline{I}} = m_I + s = d+1.$$

And (d) follows from the above (b$'$) and (c). Hence

$$H^1(\mathbf{P}^2, \overline{\mathcal{I}}(d)) = H^1(S, \mathcal{O}(d, (m_i)_{i=1}^{r+s})) = 0,$$

where $\overline{\mathcal{I}} = \prod_{i=1}^{r+s} \mathfrak{m}_{P_i}^{m_i}$ ($\mathfrak{m}_{P_i}$ is the maximal ideal of P_i). In particular, we have an exact sequence,

$$0 \to H^0(\mathbf{P}^2, \overline{\mathcal{I}}(d)) \to H^0(\mathbf{P}^2, \mathcal{O}(d)) \to \mathcal{O}/\overline{\mathcal{I}} \to 0.$$

So there exists a curve $\tilde{C}$ of degree d with multiplicity $\geq m_i$ at P_i $(i = 1, \ldots, r+s-1)$ such that $\tilde{C} \not\ni P_{r+s}$. Note that $\tilde{C}$ is transversal to L and the multiplicity of $\tilde{C}$ at P_i $(i \in I)$ is m_i since $m_{\overline{I} \setminus \{r+s\}} = d$.

Let C be a general member in the linear system generated by $C_{d-1} + L$ and $\tilde{C}$. Considering the construction of $C_{d-1} + L$ and $\tilde{C}$ and Bertini's Theorem, we have the fact that C is in $S_d(P, m)$ and transversal to L by the proof of [2, Lemma 3.3.1]. We have only to prove that the general C is transversal to L_{ij} which is not L. It is obvioius because the special member $C_{d-1} + L$ is transversal to such lines.

Case B: $m_I \leq d-2$ and there exist $p, q \in [1, r]$ $(p \neq q)$ such that $t_{d-1} \leq m_{\{p,q\}}$.

We may assume $m_i \leq m_p$ for all i, and $m_i \leq m_q$ for all $i \neq p$. If $p, q \in I$, then we are in *Case* A as in the proof of Thoerem 1. Thus we may assume $p \notin I$ or $q \notin I$.

We define $m''_i \in \mathbf{Z}_{\geq 0}$ $(1 \leq i \leq r)$ by

$$m''_i = \begin{cases} m_i - 1 & (i = p \text{ or } q), \\ m_i & (\text{otherwise}). \end{cases}$$

We omit i if $m''_i = 0$ and renumber the remaining m''_i's. We have $m'' \neq (d-1)$ since m'' has at least $r-2$ (≥ 2) components $m''_i > 0$. Then there exists a curve $C_{d-1} \in S_{d-1}(P, m'')$ transversal to L and L_{ij} for all i, j.

Let $s = d+1-m_{\{p,q\}}$ (> 0) and $P_{r+1}, \ldots, P_{r+s}$ be s distinct points on L_{pq} outside C_{d-1} and $m_{r+1} = \cdots = m_{r+s} = 1$. Let $\overline{\mathcal{I}} = \prod_{i=1}^{r+s} \mathfrak{m}_{P_i}^{m_i}$. We claim $H^1(\mathbf{P}^2, \overline{\mathcal{I}}(d)) = 0$. First $H^1(S, \mathcal{O}(d-1, (m''_i)_{i=1}^r)) = 0$ holds, since $L, d-1, m''_1, \ldots, m''_r$ satisfy (a)–(d) of Theorem 1. By $L'_{pq} \sim H - E_p - E_q - \sum_{i=r+1}^{r+s} E_i$, we have

$$\mathcal{O}(d, (m_i)_{i=1}^{r+s}) \simeq \mathcal{O}(d-1, (m''_i)_{i=1}^r)(L'_{pq}),$$

$$\Big(dH - \sum_{i=1}^{r+s} m_i E_i\Big) L'_{pq} = d - m_{\{p,q\}} - s = -1,$$

and $L'_{pq} \simeq \mathbf{P}^1$. Hence the claim

$$H^1(\mathbf{P}^2, \overline{\mathcal{I}}(d)) = H^1(S, \mathcal{O}(d, (m_i)_{i=1}^{r+s})) = 0$$

holds by Lemma 4. Then there exists a curve $\tilde{C}$ of degree d with multiplicity at least m_i at P_i $(1 \leq \forall i \leq r+s-1)$ such that $\tilde{C} \not\ni P_{r+s}$. A general member C in the linear system generated by $C_{d-1} + L$ and $\tilde{C}$ is in $S_d(P, m)$ and transversal to L and L_{ij} for all i, j.

Case C: $m_I = d-1$ and there exist $p, q \notin I$ $(p \neq q)$ such that $m_{\{p,q\}} = d$.

This can be done similarly to *Case* B.

Case D: otherwise.

If $m_{[1,r]} \leq d$ then $f(C) \in S_d(P, m)$ is transversal to L and L_{ij} for all i, j where C is a general member of $|\mathcal{O}(d, m)|$ by Lemma 5 and $d^2 - \sum_{i=1}^r m_i^2 > 0$. We may assume $m_{[1,r]} \geq d+1$. There exists a subset $J(\supset I)$ of $[1, r]$ such that $m_J \leq t_{d-1} - 1$ and $|J| = r-1$ by *Case* D of Theorem 1. This theorem holds for $I(L) = J$ by Lemma 5 and $d^2 - \sum_{i=1}^r m_i^2 > 0$. Hence the original theorem holds by the argument in the proof of [2, Lemma 3.3.1] since $H^1(S, \mathcal{O}(d, m)) = 0$ by Theorem 1 and the upper semi-continuity of h^1. The proof is complete. □

Like the previous section, we have two corollaries. Their proofs are obvious in view of the proofs of Theorem 9 and corollaries in Section 2.

When all the points are in general position, the existence theorem needs fewer conditions.

Corollary 11. *Let d, m, $M(d)$ be as in Theorem* 1. *Let $P_1, \ldots, P_r \in \mathbf{P}^2$ be r distinct points in general position. If d, m satisfy the following conditions* (a) *and* (b), *then there is a curve in $S_d(P, m)$ transversal to L and L_{ij} for all i, j.*

(a) $\sum_{i=1}^r m_i(m_i+1)/2 \leq M(d-1)$,
(b) $m_{\{i,j\}} \leq d$ $(1 \leq \forall i < \forall j \leq r)$.

When we add the upper bound $m_i \leq m_0$ $(1 \leq i \leq r)$, we can replace $M(d-1)$ with $M'(d-1)$ in Theorem 9. Like Corollary 8, we obtain Ballico's result [1].

Corollary 12. *Let d, m, P, L, I be as in Theorem* 1. *Let $m_i \leq m_0$ $(1 \leq \forall i \leq r)$. Let $M'(d)$ be as in Corollary* 8. *If d, m satisfy the following conditions* (a) *and* (b), *then there is a curve in $S_d(P, m)$ transversal to L and L_{ij} for all i, j.*

(a) $\sum_{i=1}^r m_i(m_i+1)/2 \leq M'(d-1)$,
(b) $m_I \leq d$.

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