Argument estimates for certain analytic functions

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Abstract: Let $p(z)$ be analytic in the open unit disk $\mathbf{U}$ with $p(0) = 1$ and $p'(0) = 0$. S. S. Miller and P. T. Mocanu (J. Math. Anal. Appl. **276** (2002)) have shown some interesting subordination theorems for such functions $p(z)$. The object of the present paper is to discuss some sufficient conditions for arguments of $p(z)$ to be $|\arg p(z)| < (\pi/2)\rho$ for $z \in \mathbf{U}$.

Key words: Analytic function; argument estimate; subordination.

1. Introduction. Let $p(z)$ be analytic in the open unit disk $\mathbf{U} = \{z \in \mathbf{C} : |z| < 1\}$ with $p(0) = 1$ and $p'(0) = 0$. For such functions $p(z)$, Miller and Mocanu [3] have shown some interesting subordination theorems.

Theorem A ([3]). For $1/2 < \rho \leqq 1$ define the function $q(z)$ by

$$q(z) = q_\rho(z) = \left(\frac{1+z}{1-z}\right)^\rho,$$

and let $t_0 \in (0, 1)$ be the unique solution of

$$t^\rho\left\{(1-\rho)t^2 \cos\left(\frac{\pi}{2}\rho\right) + t\sin\left(\frac{\pi}{2}\rho\right) - (1-\rho)\cos\left(\frac{\pi}{2}\rho\right)\right\} + t^2 - 1 = 0.$$

If $p(z)$ is analytic in $\mathbf{U}$, with $p(0) = 1, p'(0) = 0$ and

$$\left|\arg\left(zp'(z) + p(z)^2 + p(z)\right)\right| < \frac{\pi}{2}(\rho+1) - \mathrm{Tan}^{-1}\left(\frac{t_0}{1+\rho-(1-\rho)t_0^2}\right),$$

then $p(z) \prec q_\rho(z)$, where the symbol "$\prec$" means the subordinations.

To discuss our problems for functions $p(z)$, we need the following lemma due to Hallenbeck and Ruscheweyh [2] which is the same as one by Fukui and Sakaguchi [1].

Lemma 1.1. *Let $p(z)$ be analytic in $|z| < R$ and $p^{(k)}(0) = 0$ $(0 \leqq k \leqq n)$. Then if $|p(z)|$ attains its maximum value on the circle $|z| = r < R$ at a point z_0, we have*

$$\frac{z_0 p'(z_0)}{p(z_0)} \geqq n + 1. \tag{1.1}$$

Applying the above lemma, we derive

Lemma 1.2. *Let $p(z)$ be analytic in $\mathbf{U}$, $p(0)=1$, $p'(0) = 0$, and let $p(z) \neq 0$ $(z \in \mathbf{U})$. If there exists a point $z_0 \in \mathbf{U}$ such that*

$$|\arg p(z)| < \frac{\pi}{2}\alpha \quad (|z| < |z_0|)$$

and

$$|\arg p(z_0)| = \frac{\pi}{2}\alpha$$

for some $\alpha > 0$, then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = i\alpha k, \tag{1.2}$$

where

$$k \geqq \left(a + \frac{1}{a}\right) \geqq 2 \quad \text{when} \quad \arg p(z_0) = \frac{\pi}{2}\alpha$$

and

$$k \leqq -\left(a + \frac{1}{a}\right) \leqq -2 \quad \text{when} \quad \arg p(z_0) = -\frac{\pi}{2}\alpha,$$

where $p(z_0)^{1/\alpha} = \pm ia$ and $a > 0$.

Proof. We use the same manner which was used by Nunokawa [4] for the proof of the lemma. Let us put

$$q(z) = p(z)^{1/\alpha}. \tag{1.3}$$

Then we see that $\mathrm{Re}\, q(z) > 0$ $(|z| < |z_0|)$, $\mathrm{Re}\, q(z_0) = 0$, $q(0) = 1$ and $q'(0) = 0$. Defining the function $\phi(z)$

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by

(1.4) $$\phi(z)=\frac{1-q(z)}{1+q(z)},$$

we have that $\phi(0)=0$, $|\phi(z)|<1$ ($|z|<|z_0|$), and $|\phi(z_0)|=1$.

In view of Lemma 1.1, we know that

(1.5) $$\frac{z_0\phi'(z_0)}{\phi(z_0)}=\frac{-2z_0q'(z_0)}{1-q(z_0)^2}=\frac{-2z_0q'(z_0)}{1+|q(z_0)|^2}\geqq 2.$$

It follows from (1.5) that

(1.6) $$-z_0q'(z_0)\geqq\left(1+|q(z_0)|^2\right)$$

and $z_0q'(z_0)$ is a negative real number. Since $q(z_0)$ is a non-vanishing pure imaginary number, we can put $q(z_0)=ia$, where a is a non-vanishing real number. We have, for $a>0$,

(1.7) $$\operatorname{Im}\left(\frac{z_0q'(z_0)}{q(z_0)}\right)=\operatorname{Im}\left(-\frac{iz_0q'(z_0)}{|q(z_0)|}\right)\geqq\left(\frac{1+a^2}{a}\right)\geqq 2$$

and, for $a<0$,

(1.8) $$\operatorname{Im}\left(\frac{z_0q'(z_0)}{q(z_0)}\right)=\operatorname{Im}\left(\frac{iz_0q'(z_0)}{|q(z_0)|}\right)\leqq-\left(\frac{1+a^2}{a}\right)\leqq -2.$$

On the other hand, it follows that

(1.9) $$\frac{z_0q'(z_0)}{q(z_0)}=\frac{1}{\alpha}\left(\frac{z_0p'(z_0)}{p(z_0)}\right).$$

This completes the proof of Lemma 1.2. □

2. Argument estimates. Our first property for argument estimates of analytic function $p(z)$ is contained in

Theorem 2.1. *Let $p(z)$ be analytic in* $\mathbf{U}$ *with $p(0)=1$ and $p'(0)=0$. If $p(z)$ satisfies*

(2.1) $$\left|\arg\left(zp'(z)+p(z)^2+\alpha p(z)\right)\right|<\pi\rho\quad (z\in\mathbf{U})$$

for some α ($\alpha>0$), ρ ($0<\rho\leqq\rho_0$), where ρ_0 ($0<\rho_0<1$) is given by

$$\tan\left(\frac{\pi}{2}\rho_0\right)=\frac{2}{\alpha}\rho_0,$$

then

(2.2) $$|\arg p(z)|<\frac{\pi}{2}\rho\quad (z\in\mathbf{U}).$$

Proof. Let a function $p(z)$ satisfy the conditions of the theorem. If there exists a point $z_0\in\mathbf{U}$ such that

$$|\arg p(z)|<\frac{\pi}{2}\rho\quad (|z|<|z_0|)$$

and

$$|\arg p(z_0)|=\frac{\pi}{2}\rho,$$

then applying Lemma 1.2, we have that

(2.3) $$\frac{z_0p'(z_0)}{p(z_0)}=i\rho k,$$

where

$$k\geqq a+\frac{1}{a}\geqq 2\quad\text{when}\quad \arg p(z_0)=\frac{\pi}{2}\rho$$

and

$$k\leqq-\left(a+\frac{1}{a}\right)\leqq -2\quad\text{when}\quad \arg p(z_0)=-\frac{\pi}{2}\rho$$

with $p(z_0)^{1/\rho}=\pm ia$ ($a>0$). It follows that, for $\arg p(z_0)=\pi/2\rho$ and $k\geqq a+1/a\geqq 2$,

(2.4) $$\begin{aligned}\arg&\left(z_0p'(z_0)+p(z_0)^2+\alpha p(z_0)\right)\\&=\arg p(z_0)\left(\frac{z_0p'(z_0)}{p(z_0)}+p(z_0)+\alpha\right)\\&=\frac{\pi}{2}\rho+\arg\left(i\rho k+a^\rho e^{i\frac{\pi}{2}\rho}+\alpha\right)\\&=\frac{\pi}{2}\rho+\operatorname{Tan}^{-1}\left(\frac{\rho k+a^\rho\sin\left(\frac{\pi}{2}\rho\right)}{\alpha+a^\rho\cos\left(\frac{\pi}{2}\rho\right)}\right).\end{aligned}$$

Since, by $0<\rho\leqq\rho_0<1$ and $k\geqq 2$,

(2.5) $$\operatorname{Tan}^{-1}\left(\frac{\rho k+a^\rho\sin\left(\frac{\pi}{2}\rho\right)}{\alpha+a^\rho\cos\left(\frac{\pi}{2}\rho\right)}\right)\geqq\operatorname{Tan}^{-1}\left(\frac{2\rho+a^\rho\sin\left(\frac{\pi}{2}\rho\right)}{\alpha+a^\rho\cos\left(\frac{\pi}{2}\rho\right)}\right)>0,$$

we define $g(a)$ by

(2.6) $$g(a)=\frac{2\rho+a^\rho\sin\left(\frac{\pi}{2}\rho\right)}{\alpha+a^\rho\cos\left(\frac{\pi}{2}\rho\right)}\quad (a>0).$$

Noting that

(2.7) $$g'(a)=\frac{\alpha\rho a^{\rho-1}\cos\left(\frac{\pi}{2}\rho\right)\left(\tan\left(\frac{\pi}{2}\rho\right)-\frac{2\rho}{\alpha}\right)}{\left(\alpha+a^\rho\cos\left(\frac{\pi}{2}\rho\right)\right)^2},$$

we define $h(\rho)$ by

(2.8) $$h(\rho)=\tan\left(\frac{\pi}{2}\rho\right)-\frac{2\rho}{\alpha}\quad (0<\rho\leqq\rho_0<1).$$

Then $h(0)=0, h(\rho_0)=0$, and

$$(2.9)\qquad h''(\rho) = \frac{\pi^2}{2}\sec^2\left(\frac{\pi}{2}\rho\right)\tan\left(\frac{\pi}{2}\rho\right) > 0.$$

This shows that $g'(a) \leqq 0$ for $a > 0$, that is, that

$$(2.10)\qquad \mathrm{Tan}^{-1}\left(\frac{\rho k + a^\rho \sin\left(\frac{\pi}{2}\rho\right)}{\alpha + a^\rho \cos\left(\frac{\pi}{2}\rho\right)}\right) \geqq \mathrm{Tan}^{-1}\left(\tan\left(\frac{\pi}{2}\rho\right)\right) = \frac{\pi}{2}\rho.$$

Therefore, we conclude that

$$(2.11)\qquad \arg\left(z_0p'(z_0) + p(z_0)^2 + \alpha p(z_0)\right) \geqq \pi\rho$$

when $\arg p(z_0) = (\pi/2)\rho$.

Similarly, for $\arg p(z_0) = -(\pi/2)\rho$ and $k \leqq -(a+1/a) \leqq -2$, we have that

$$\begin{aligned}(2.12)\quad &\arg\left(z_0p'(z_0) + p(z_0)^2 + \alpha p(z_0)\right)\\ &= -\frac{\pi}{2}\rho + \arg\left(i\rho k + a^\rho e^{-i\frac{\pi}{2}\rho} + \alpha\right)\\ &= -\frac{\pi}{2}\rho + \mathrm{Tan}^{-1}\left(\frac{\rho k - a^\rho \sin\left(\frac{\pi}{2}\rho\right)}{\alpha + a^\rho \cos\left(\frac{\pi}{2}\rho\right)}\right)\\ &\leqq -\frac{\pi}{2}\rho + \mathrm{Tan}^{-1}\left(\frac{-2\rho - a^\rho \sin\left(\frac{\pi}{2}\rho\right)}{\alpha + a^\rho \cos\left(\frac{\pi}{2}\rho\right)}\right)\\ &= -\frac{\pi}{2}\rho - \mathrm{Tan}^{-1}\left(\frac{2\rho + a^\rho \sin\left(\frac{\pi}{2}\rho\right)}{\alpha + a^\rho \cos\left(\frac{\pi}{2}\rho\right)}\right)\\ &\leqq -\frac{\pi}{2}\rho - \frac{\pi}{2}\rho = -\pi\rho.\end{aligned}$$

Thus, for such a point $z_0 \in \mathbf{U}$. we see that

$$(2.13)\qquad |\arg\left(z_0p'(z_0) + p(z_0)^2 + \alpha p(z_0)\right)| \geqq \pi\rho,$$

which contradicts our condition for $p(z)$.

Consequently, we conclude that

$$|\arg p(z)| < \frac{\pi}{2}\rho \quad (z \in \mathbf{U}).$$

□

Example 2.1. Let us consider the function $p(z)$ defined by

$$p(z) = 1 + \frac{1}{5}z^2.$$

Then we see that

$$zp'(z) + p(z)^2 + \frac{1}{2}p(z) = \frac{3}{2} + \frac{9}{10}z^2 + \frac{1}{25}z^4.$$

Letting $\alpha = 1/2$ and

$$\rho = \frac{1}{\pi}\mathrm{Sin}^{-1}\left(\frac{19}{30}\right)$$

in Theorem 2.1, we have that

$$\left|\arg\left(zp'(z) + p(z)^2 + \frac{1}{2}p(z)\right)\right| < \pi\rho = \mathrm{Sin}^{-1}\left(\frac{19}{30}\right)$$

and

$$|\arg p(z)| < \mathrm{Sin}^{-1}\left(\frac{1}{5}\right) < \frac{\pi}{2}\rho.$$

If we take $\alpha = 1$ in Theorem 2.1, then

Corollary 2.1. *Let $p(z)$ be analytic in $\mathbf{U}$ with $p(0) = 1$ and $p'(0) = 0$. If $p(z)$ satisfies*

$$(2.14)\qquad |\arg\left(zp'(z) + p(z)^2 + p(z)\right)| < \pi\rho \quad (z \in \mathbf{U})$$

for some ρ $(0 < \rho \leqq 1/2)$, then

$$(2.15)\qquad |\arg p(z)| < \frac{\pi}{2}\rho \quad (z \in \mathbf{U}).$$

Remark 2.1. (1) If $\alpha = 4/5$, then $0 < \rho \leqq \rho_0$ and $0.647873 < \rho_0 < 0.647874$.

(2) If $\alpha = 1/2$, then $0 < \rho \leqq \rho_0$ and $0.809251 < \rho_0 < 0.809252$.

(3) If $\alpha = 1/3$, then $0 < \rho \leqq \rho_0$ and $0.880966 < \rho_0 < 0.880967$.

(4) If $\alpha = 1/4$, then $0 < \rho \leqq \rho_0$ and $0.913417 < \rho_0 < 0.913418$.

(5) If $\alpha = 1.1$, then $0 < \rho \leqq \rho_0$ and $0.401248 < \rho_0 < 0.401249$.

(6) If $\alpha = 1.2$, then $0 < \rho \leqq \rho_0$ and $0.262943 < \rho_0 < 0.262944$.

(7) If $\alpha = 1.3$, then there is no $\rho_0 > 0$ such that $\tan(\pi/2)\rho_0 = (2/\alpha)\rho$. Thus we see that $0 < \alpha < 1.3$ in Theorem 2.1.

Next, we derive

Theorem 2.2. *Let $p(z)$ be analytic in $\mathbf{U}$ with $p(0) = 1$ and $p'(0) = 0$. If $p(z)$ satisfies*

$$(2.16)\qquad \left|\arg\left(zp'(z) + p(z)^2 + \alpha p(z)\right)\right| < \frac{\pi}{2}\rho + \mathrm{Tan}^{-1}\left(\frac{2\rho}{\alpha}\right) \quad (z \in \mathbf{U})$$

for some α $(\alpha > 0)$, ρ $(\rho_0 \leqq \rho < 1)$, where ρ_0 $(0 < \rho_0 < 1)$ is given by $\tan(\pi/2)\rho_0 = (2/\alpha)\rho_0$, then

$$(2.17)\qquad |\arg p(z)| < \frac{\pi}{2}\rho \quad (z \in \mathbf{U}).$$

Proof. Using the same technique as in the proof of Theorem 2.1, we know that

$$\mathrm{Tan}^{-1}\left(\frac{2\rho + a^\rho \sin\left(\frac{\pi}{2}\rho\right)}{\alpha + a^\rho \cos\left(\frac{\pi}{2}\rho\right)}\right)$$

is increasing for $a > 0$. Thus, we obtain

(2.18) $$\left|\arg\left(z_0p'(z_0)+p(z_0)^2+\alpha p(z_0)\right)\right| \geqq \frac{\pi}{2}\rho+\mathrm{Tan}^{-1}\left(\frac{2\rho}{\alpha}\right)$$

for $z_0 \in \mathbf{U}$ such that

$$|\arg p(z)| < \frac{\pi}{2}\rho \quad (|z|<|z_0|)$$

and

$$|\arg p(z_0)| = \frac{\pi}{2}\rho.$$

This contradicts our condition of the theorem. Therefore,

$$|\arg p(z)| < \frac{\pi}{2}\rho \quad (z \in \mathbf{U}).$$

□

Letting $\alpha = 1$ in Theorem 2.2, we obtain

Corollary 2.2. *Let $p(z)$ be analytic in $\mathbf{U}$ with $p(0)=1$ and $p'(0)=0$. If $p(z)$ satisfies*

(2.19) $$\left|\arg\left(zp'(z)+p(z)^2+p(z)\right)\right| < \frac{\pi}{2}\rho+\mathrm{Tan}^{-1}(2\rho) \quad (z\in\mathbf{U})$$

for some ρ $(1/2 \leqq \rho < 1)$, then

(2.20) $$|\arg p(z)| < \frac{\pi}{2}\rho \quad (z \in \mathbf{U}).$$

Finally, we note that

Remark 2.2. (1) If $\alpha = 4/5$, then $0<\rho\leqq\rho_0$ and $0.647873<\rho_0<0.647874$.

(2) If $\alpha = 1/2$, then $0<\rho\leqq\rho_0$ and $0.809251<\rho_0<0.809252$.

(3) If $\alpha = 1/3$, then $0<\rho\leqq\rho_0$ and $0.880966<\rho_0<0.880967$.

(4) If $\alpha = 1/4$, then $0<\rho\leqq\rho_0$ and $0.913417<\rho_0<0.913418$.

(5) If $\alpha = 1.1$, then $0<\rho\leqq\rho_0$ and $0.401248<\rho_0<0.401249$.

(6) If $\alpha = 1.2$, then $0<\rho\leqq\rho_0$ and $0.262943<\rho_0<0.262944$.

(7) If $\alpha = 1.3$, then there is no $\rho_0>0$ such that $\tan(\pi/2)\rho_0 = (2/\alpha)\rho$. Thus we see that $0<\alpha<1.3$ in Theorem 2.2.

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