Strong unique continuation property of two-dimensional Dirac equations with Aharonov-Bohm fields

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Abstract: We study the unique continuation property of two-dimensional Dirac equations with Aharonov-Bohm fields. Some results for the unperturbed Dirac operator are given by De Carli-Ōkaji [2]. We are interested in the problem how the singularity of Aharonov-Bohm fields at the origin influences the unique continuation property.

Key words: Aharonov-Bohm effect; Dirac operator; unique continuation.

1. Introduction. It is well known that, if any harmonic function $u(x)$ in a domain $\Omega \subset \mathbf{R}^n$ satisfies

$$\partial_x^\alpha u(x_0) = 0$$

for all multi-indices α at a point $x_0 \in \Omega$, then $u(x)$ vanishes identically in Ω. Recently, it is shown by Grammatico [3] that, if Ω contains the origin and $u \in W^{2,2}_{\mathrm{loc}}(\Omega)$ (Sobolev space) satisfies

$$(1) \qquad |\Delta u| \le \frac{M}{|x|^2}|u(x)| + \frac{C}{|x|}|\nabla u|$$

(a.e. on Ω) with $M > 0$ and $0 < C < 1/\sqrt{2}$, and

$$(2) \qquad \lim_{\varepsilon\to+0} \varepsilon^{-\ell} \int_{|x|<\varepsilon} |u|^2 dx = 0,$$

then $u(x)$ vanishes identically in Ω (one can see some related works in the References of Grammatico [3]). Then we say that the inequality (1) has the strong unique continuation property. If $u(x)$ satisfies (2), $u(x)$ is said to vanish of infinite order at the origin, or to be flat at the origin. We can not expect the strong unique continuation property for every $C > 0$. For Alinhac-Baouendi [1] shows that, if $C > 1$, there is a non-trivial complex-valued function $v \in C^\infty(\mathbf{R}^2)$, which is flat at the origin satisfying $\operatorname{supp} v = \mathbf{R}^2$ and (1) with $M = 0$ (see also Pan-Wolff [6]).

For corresponding problems to the Dirac operator

$$L_0 = \sum_{j=1}^n \alpha_j p_j \quad \left(p_j = \frac{1}{\sqrt{-1}}\frac{\partial}{\partial x_j}, \quad n \ge 2\right),$$

where α_j are $N \times N$ Hermitian matrices satisfying $\alpha_j\alpha_k + \alpha_k\alpha_j = 2\delta_{jk}I_N$ ($N = 2^{[(n+1)/2]}$), De Carli-Ōkaji [2] shows that, if a positive constant $C < 1/2$, then the inequality

$$(3) \qquad |L_0 u| \le \frac{C}{|x|}|u| \ \text{ a.e. on } \ \Omega \quad (u \in W^{1,2}_{\mathrm{loc}}(\Omega)^N)$$

has the strong unique continuation property, where $|u| = \sqrt{|u_1|^2 + |u_2|^2}$ (see also Kalf-Yamada [4] and Ōkaji [5]). The restriction on $C < 1/2$ is needed to treat the angular momentum term (spin-orbit term) but the radial part of L_0. As is also pointed out by De Carli-Ōkaji [2], the counter example by Alinhac-Baouendi [1] implies that a certain restriction on the constant C in (3) is also necessary. In fact, if we set

$$u_1 := \partial\, u = (\partial_1 - i\partial_2)v, \quad u_2 := \bar\partial\, u = (\partial_1 + i\partial_2)v,$$

then we can see that u_1 and $u_2 \not\equiv 0$ are flat at the origin satisfying (1) with the same constant $C > 1$ (cf. Corollary below). It is an open problem what happens for $1/2 \le C \le 1$. In this note we investigate the strong unique continuation property for 2-dimensional Dirac operators with Aharonov-Bohm efect, which is one of singular magnetic fields at the origin, and give a perturbation to the spin-orbit term. Our proof is given along the same line as in De Carli-Ōkaji [2] and Kalf-Yamada [4].

2. The result. Let us consider 2-dimensional Dirac operators with Aharonov-Bohm fields

$$L_\beta := \sigma \cdot D = \sigma_1 D_1 + \sigma_2 D_2,$$

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where

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$
$$D_j := p_j - b_j(x) = -i\frac{\partial}{\partial x_j} - b_j(x),$$
$$b_1(x) := -\beta\,\frac{x_2}{|x|^2}, \quad b_2(x) := \beta\,\frac{x_1}{|x|^2},$$

and β is a real number. Such a magnetic field has a delicate singularity at the origin in spectral theory (see, e.g., Tamura [7]).

Put $\tilde\beta := \beta - [\beta]$, where $[\cdot]$ is Gauss's symbol.

Theorem 1. *Let Ω be a connected open set in $\mathbf{R}^2$ containing the origin. If $u \in W^{1,2}_{\mathrm{loc}}(\Omega)^2$ is flat at the origin and*

$$|L_\beta\, u| \le \frac{C_0}{|x|}\,|u| \tag{4}$$

a.e. on Ω for a positive constant $C_0 < \gamma(\beta)$ with

$$\gamma(\beta) := \begin{cases} \dfrac{1-2\tilde\beta}{2} & \left(0 \le \tilde\beta < \dfrac14\right), \\ \tilde\beta & \left(\dfrac14 \le \tilde\beta < \dfrac12\right), \\ 1-\tilde\beta & \left(\dfrac12 \le \tilde\beta < \dfrac34\right), \\ \dfrac{2\tilde\beta-1}{2} & \left(\dfrac34 \le \tilde\beta < 1\right), \end{cases}$$

then u vanishes identically on Ω.

Corollary. *Let $S_\beta := D_1^2 + D_2^2$ be the Schrödinger operator. Let Ω be an open set containing the origin. If $v \in W^{2,2}_{\mathrm{loc}}(\Omega)$ is flat at the origin satisfying*

$$|S_\beta v| \le \frac{C_0}{|x|}\,|Dv| \tag{5}$$

a.e. on Ω for a positive constant $C_0 < \gamma(\beta)$, then v vanishes identically on Ω, where $|Dv| := \sqrt{|D_1 v|^2 + |D_2 v|^2}$.

For the proof of Corollary, let us put $u_1 := (D_1 - iD_2)v$ and $u_2 := (D_1 + iD_2)v$. Since v is flat at the origin, we can show that $D_1 v$ and $D_2 v$ are flat at the origin by using (5). Therefore, u_1 and u_2 are flat at the origin and satisfy

$$D_1 v = \frac{u_1+u_2}{2}, \quad D_2 v = -\frac{u_1-u_2}{2i},$$

$$D_1D_2 v = D_2 D_1 v.$$

Moreover, we have

$$\begin{aligned} |L_\beta u| &= \sqrt2\,|(D_1^2+D_2^2)v| \le \frac{\sqrt2\,C_0}{|x|}\,|Dv| \\ &= \frac{C_0}{\sqrt2\,|x|}\sqrt{|u_1-u_2|^2 + |u_1+u_2|^2} \\ &= \frac{C_0}{|x|}\,|u|, \end{aligned}$$

which gives from Theorem 1 that $u_1 = u_2 \equiv 0$ and $(\partial v/\partial r) \equiv 0$ in Ω. Since v is flat at the origin, we have $v \equiv 0$.

3. Proofs. Here we introduce some notations. Let

$$D_r := \sum_{j=1}^2 \frac{x_j}{r} D_j, \qquad \sigma_r = \sum_{j=1}^2 \frac{x_j}{r}\sigma_j,$$

$$\begin{aligned} S &:= \frac12 - i\sigma_1\sigma_2(x_1D_2 - x_2D_1) \\ &= \frac12 + \sigma_3(x_1p_2 - x_2p_1 - \beta), \end{aligned}$$

where

$$\sigma_3 := -i\sigma_1\sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The spin-orbit operator S is written by polar coordinates $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$ as

$$S = \begin{pmatrix} \dfrac12 - \beta - i\dfrac{\partial}{\partial\theta} & 0 \\ 0 & \dfrac12 + \beta + i\dfrac{\partial}{\partial\theta} \end{pmatrix}, \tag{6}$$

which can be regarded as a self-adjoint operator on $L^2(S^1)^2$. Then we have

$$\sigma\cdot D = \sigma_r\left(D_r + \frac{i}{r}S\right), \quad \sigma_r^2 = I, \tag{7}$$

$$\sigma_r D_r = D_r\sigma_r, \quad \sigma_r S = -S\sigma_r, \quad D_r S = S D_r, \tag{8}$$

$$D_r^2 \ge \frac{1}{4r^2} \tag{9}$$

on $C_0^\infty(\mathbf{R}^2\setminus\{0\})^2$. The last inequality can be shown by a commutator relation

$$\left[D_r, \frac1r\right] = \frac{i}{r^2}.$$

Lemma 2. *For a real number m we put*

$$A := \sigma\cdot D - i\frac{m}{r}\sigma_r.$$

Then we have

$$A^*A \ge \frac{1}{r^2}\left(S - m - \frac12\right)^2 \tag{10}$$

on $C_0^\infty(\mathbf{R}^2 \setminus \{0\})^2$*, and the spectrum* $\sigma(S)$ *consists of discrete eigenvalues*

$$\left\{ n + \frac{1}{2} \pm \beta \mid n \in \mathbf{Z} \right\}. \tag{11}$$

Proof. The properties (7), (8) and (9) give

$$\begin{aligned} A^*A &= \left[\sigma_r \left(D_r + \frac{i}{r} S \right) + \frac{im}{r} \sigma_r \right] \\ &\quad \cdot \left[\sigma_r \left(D_r + \frac{i}{r} S \right) - \frac{im}{r} \sigma_r \right] \\ &= \left[D_r - \frac{i}{r}(S - m) \right] \left[D_r + \frac{i}{r}(S - m) \right] \\ &= D_r^2 - \frac{1}{4r^2} + \frac{1}{r^2} \left(S - m - \frac{1}{2} \right)^2 \\ &\geq \frac{1}{r^2} \left(S - m - \frac{1}{2} \right)^2, \end{aligned}$$

which shows (10). Since S has a complete orthonormal eigenfunctions in $L^2(S^1)^2$,

$$\frac{1}{\sqrt{2\pi}} \begin{pmatrix} e^{in\theta} \\ 0 \end{pmatrix}, \quad \frac{1}{\sqrt{2\pi}} \begin{pmatrix} 0 \\ e^{-in\theta} \end{pmatrix} \quad (n \in \mathbf{Z}),$$

we obtain (11). □

Lemma 3. *There exists a sequence of positive numbers* m_j $(j = 1, 2, \cdots)$ *with* $m_j \to \infty$ *as* $j \to \infty$ *such that*

$$\|r^{-m_j}(\sigma \cdot D)u\| \geq \gamma(\beta)\|r^{-m_j-1}u\|$$

for any $u \in W^{1,2}(\mathbf{R}^2)^2$ *whose support does not include a neighborhood of the origin, where* $\gamma(\beta)$ *is what is defined in Theorem* 1.

Proof. Let $\varphi \in C_0^\infty(\mathbf{R}^2 \setminus \{0\})^2$. In view of Lemma 2 we have

$$\begin{aligned} &\int_{\mathbf{R}^2} r^{-2m} |\sigma \cdot D\varphi|^2 dx \\ &= \int_{\mathbf{R}^2} \left| A \left(r^{-m} \varphi \right) \right|^2 dx \\ &\geq \min_{n \in \mathbf{Z}} |n \pm \beta - m|^2 \int_{\mathbf{R}^2} r^{-2m-2} |\varphi|^2 dx \end{aligned}$$

for any $\varphi \in C_0^\infty(\mathbf{R}^2 \setminus \{0\})^2$ and $m \in \mathbf{R}$. Seeing the definition of $\gamma(\beta)$ in Theorem 1, we can find a sequence $m_j \to \infty$ such that

$$\min_{n \in \mathbf{Z}} |n \pm \beta - m_j|^2 = \gamma(\beta).$$

For a given $u \in W^{1,2}(\mathbf{R}^2)^2$ whose support does not include a neighborhood of the origin, there exists a sequence $\{\varphi_j\}_{j=1,2,\cdots} \subset C_0^\infty(\mathbf{R}^2 \setminus \{0\})^2$ such that $\varphi_j \to u$ in $W^{1,2}(\mathbf{R}^2)$ $(j \to \infty)$, which completes the proof. □

Lemma 3 yields the following

Lemma 4. *Suppose that* $u \in W^{1,2}_{\mathrm{loc}}(\Omega)^2$ *is flat at the origin with* (4)*. Let* $B_{R_0} := \{x \in \mathbf{R}^2 : |x| < R_0\} \subset \Omega$*. For any* $R_1 < R_0$ *there exists a positive constant* $C_1 = C_1(R_0, R_1)$ *independent of* m_j *such that*

$$\begin{aligned} &[\gamma(\beta)^2 - C_0^2] \int_{B_{R_1}} r^{-2m_j-2} |u|^2 \, dx \\ &\leq 2C_0^2 \int_{R_1 < |x| < R_0} r^{-2m_j-2} |u|^2 \, dx \\ &\quad + C_1 \int_{R_1 < |x| < R_0} r^{-2m_j} |u|^2 \, dx, \end{aligned} \tag{12}$$

where m_j *is the one given in Lemma* 3.

Proof. Fix $0 < R_1 < R_0$ and take $\delta > 0$ and a smooth function $\chi_\delta \in C_0^\infty(0, R_0)$ such that

$$\chi_\delta(r) = \begin{cases} 1 & (\delta \leq r \leq R_1) \\ 0 & (r \leq \delta/2) \end{cases}$$

and

$$|\chi'_\delta(r)| \leq \begin{cases} C_2 \delta^{-1} & (\delta/2 \leq r \leq \delta) \\ C_2 & (R_1 \leq r \leq R_0) \end{cases}$$

for a positive constant C. Then Lemma 3 and the condition (4) yield

$$\begin{aligned} &\gamma(\beta)^2 \int_{\delta \leq r \leq R_1} r^{-2m_j-2} |u|^2 \, dx \\ &\leq \gamma(\beta)^2 \int r^{-2m_j-2} |\chi_\delta u|^2 \, dx \\ &\leq \int |r^{-2m_j} (\sigma \cdot D)(\chi_\delta u)|^2 \, dx \\ &\leq 2 \int_{\delta/2 \leq r \leq \delta} r^{-2m_j} \left[C_2^2 \delta^{-2} + C_0^2 r^{-2} \right] |u|^2 \, dx \\ &\quad + C_0^2 \int_{\delta \leq r \leq R_1} r^{-2m_j-2} |u|^2 \, dx \\ &\quad + 2 \int_{R_1 \leq r \leq R_2} r^{-2m_j} \left[C_2^2 + C_0^2 r^{-2} \right] |u|^2 \, dx. \end{aligned} \tag{13}$$

Since u is flat at the origin, the last three integrals tend to zero if $\delta \to 0$. Therefore we have (12) with $C_1 = 2C_2^2$. □

Proof of Theorem 1. Let $B_{R_0} \subset \Omega$ and take $0 < R_2 < R_1 < R_0$. In view of (12) we have

$$[\gamma(\beta)^2 - C_0^2]\left(\frac{R_1}{R_2}\right)^{2m_j}\int_{B_{R_2}}\frac{|u|^2}{r^2}\,dx$$
$$\le [\gamma(\beta)^2 - C_0^2]R_1^{2m_j}\int_{B_{R_1}} r^{-2m_j-2}|u|^2\,dx$$
$$\le 2C_0^2R_1^{2m_j}\int_{R_1<|x|<R_0} r^{-2m_j-2}|u|^2\,dx$$
$$+ C_1R_1^{2m_j}\int_{R_1<|x|<R_0} r^{-2m_j}|u|^2\,dx$$
$$\le 2C_0^2\int_{R_1<|x|<R_0}\frac{|u|^2}{r^2}\,dx$$
$$+ C_1\int_{R_1<|x|<R_0}|u|^2\,dx.$$

Making $m_j \to \infty$, we have $u \equiv 0$ in B_{R_2}. Since R_1 and R_2 are arbitrary, we have $u \equiv 0$ in B_{R_0}.

Assume that there is $x_0 \in \Omega$ with $|x_0| = R_0$. The condition (3) yields

$$|L_0u| \le \frac{C_0 + |\beta|}{|x|}\,|u| \quad \text{in} \quad \Omega.$$

Set $x_\varepsilon = (1-\varepsilon)x_0$ for $0 < \varepsilon < R_0$. If

$$0 < \rho < \frac{R_0 - \varepsilon}{1 + 2(C_0 + |\beta|)},$$

then we can find a positive constant $C' < 1/2$ such that

$$|L_0u| \le \frac{C'}{|x - x_\varepsilon|}\,|u| \quad \text{in} \quad \Omega \cap B_\rho(x_\varepsilon),$$

where $B_\rho(x_\varepsilon)$ is the open ball with radius ρ and center x_ε. This fact implies, by De Carli-Ōkaji [2],

$$u \equiv 0 \quad \text{in} \quad \Omega \cap B_{R_1},$$

where $R_1 := R_0\left[1 + \{2(C_0 + |\beta|) + 1\}^{-1}\right]$. By repeating this procedure we have $u \equiv 0$ in Ω. □

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