Duplication formulas in triple trigonometry

By Nobushige KUROKAWA*) and Masato WAKAYAMA**)

(Communicated by Heisuke HIRONAKA, M. J. A., Oct. 14, 2003)

Abstract: We study the duplication formulas of triple sine and cosine functions from various expressions point of view.

Key words: Riemann zeta function; special values; multiple trigonometric function; duplication formula.

1. Introduction. Triple trigonometric functions

$$\mathcal{S}_3(x) = e^{\frac{x^2}{2}} \prod_{n=1}^{\infty} \left\{ \left(1 - \frac{x^2}{n^2}\right)^{n^2} e^{x^2} \right\}$$

and

$$\mathcal{C}_3(x) = \prod_{n=1}^{\infty} \left\{ \left(1 - \frac{x^2}{\left(n-\frac{1}{2}\right)^2}\right)^{\left(n-\frac{1}{2}\right)^2} e^{x^2} \right\}$$

are interesting special functions. For example, the special value $\mathcal{S}_3(1/2)$ gives the expression

$$\zeta(3) = \frac{8\pi^2}{7} \log\left(\mathcal{S}_3\left(\frac{1}{2}\right)^{-1} 2^{\frac{1}{4}}\right)$$

to the famous mysterious zeta-value

$$\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}$$

as proved in [KW1].

We therefore want to know the arithmetic nature of special values of $\mathcal{S}_3(x)$ and $\mathcal{C}_3(x)$. Since these trigonometric functions are generalizations of the usual trigonometric functions

$$\mathcal{S}_1(x) = 2\pi x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right) = 2\sin(\pi x)$$

and

$$\mathcal{C}_1(x) = 2 \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{\left(n-\frac{1}{2}\right)^2}\right) = 2\cos(\pi x),$$

it would be natural to study the analogue of the following well-known duplication formulas:

(A) $\mathcal{S}_1(2x) = \mathcal{S}_1(x)\mathcal{C}_1(x)$,

(B) $\mathcal{S}_1(2x) = \mathcal{S}_1(x)\mathcal{S}_1\left(x + \frac{1}{2}\right)$,

(C) $\mathcal{S}_1(2x) = \Phi(\mathcal{S}_1(x))$ for $\Phi(u) = \pm u\sqrt{4 - u^2}$,
$\mathcal{C}_1(2x) = \Psi(\mathcal{C}_1(x))$ for $\Psi(u) = u^2 - 2$.

Our first results are generalizations of (A) and (B) which we call multiplicative duplication formulas.

Theorem 1.1. *The multiplicative duplication formulas of $\mathcal{S}_3(x)$ hold*:

(A) $\mathcal{S}_3(2x) = \mathcal{S}_3(x)^4\mathcal{C}_3(x)^4$,

(B) $$\mathcal{S}_3(2x) = \exp\left(\frac{7\zeta(3)}{2\pi^2}\right)\mathcal{S}_3(x)^4\mathcal{S}_3\left(x+\frac{1}{2}\right)^4 \times \mathcal{S}_2\left(x+\frac{1}{2}\right)^{-4}\mathcal{S}_1\left(x+\frac{1}{2}\right).$$

Also the analogue of the duplication formulas of type (C) we seek are of the form respectively given by

$$\mathcal{S}_3(2x) = \Phi(\mathcal{S}_3(x))$$

and

$$\mathcal{C}_3(2x) = \Psi(\mathcal{C}_3(x))$$

where $\Phi(u)$ and $\Psi(u)$ belong in $\overline{\mathbf{Q}}[[u]]$.

If we have such formulas, from the fact $\mathcal{S}_3(1) = 0$ we have $\Phi(\mathcal{S}_3(1/2)) = 0$, which might imply the algebraicity of the value

$$\mathcal{S}_3\left(\frac{1}{2}\right) = 2^{\frac{1}{4}} \exp\left(-\frac{7\zeta(3)}{8\pi^2}\right).$$

Moreover, from the fact $\Psi\left(\mathcal{C}_3(1/4)\right) = \mathcal{C}_3(1/2) = 0$ we would have the algebraicity of

$$\mathcal{C}_3\left(\frac{1}{4}\right) = 2^{\frac{1}{32}} \exp\left(\frac{21\zeta(3)}{64\pi^2} + \frac{L(2,\chi_{-4})}{4\pi}\right)$$

where

$$L(s,\chi_{-4}) = \sum_{n=1}^{\infty} \frac{\chi_{-4}(n)}{n^s} = \sum_{n=1}^{\infty} \frac{(-1)^{\frac{n-1}{2}}}{n^s}$$

2000 Mathematics Subject Classification. 11M06.

*) Department of Mathematics, Tokyo Institute of Technology, 2-12-1, Oh-okayama, Meguro-ku, Tokyo 152-8550.

**) Faculty of Mathematics, Kyushu University, 6-10-1, Hakozaki, Higashi-ku, Fukuoka 812-8581.

is the Dirichlet L-function for the non-trivial Dirichlet character χ_{-4} modulo 4.

The following two theorems are results in the direction of (C).

Theorem 1.2. *There exists a power series*

$$\Phi(u) = -16u^4 + 8u^6 + \cdots$$

belonging to $\mathbf{R}[[u]]$ *which satisfies*

$$\mathcal{S}_3(2x) = \Phi(\mathcal{S}_3(x))$$

around $x = 1$.

Theorem 1.3. *There exists a power series*

$$\Psi(u) = -16 + 32\exp\Big(-\frac{7\zeta(3)}{\pi^2}\Big)u^2 + \cdots$$

belonging to $\mathbf{R}[[u]]$ *which satisfies*

$$\mathcal{C}_3(2x)^4 = \Psi(\mathcal{C}_3(x)^4)$$

around $x = 1/2$.

2. Multiple trigonometric functions. We recall here basic multiple trigonometry from [K1, KKo, KOW]. Multiple trigonometric functions of degree $r \geq 2$ are consisting of the multiple sine function

$$\mathcal{S}_r(x) = e^{\frac{x^{r-1}}{r-1}} \prod_{n=1}^{\infty} \Big\{ P_r\Big(\frac{x}{n}\Big) P_r\Big(-\frac{x}{n}\Big)^{(-1)^{r-1}} \Big\}^{n^{r-1}}$$

and the multiple cosine function

$$\mathcal{C}_r(x) = \prod_{n=1}^{\infty} \Big\{ P_r\Big(\frac{x}{n-\frac{1}{2}}\Big) P_r\Big(-\frac{x}{n-\frac{1}{2}}\Big)^{(-1)^{r-1}} \Big\}^{(n-\frac{1}{2})^{r-1}},$$

where

$$P_r(u) = (1-u)\exp\left(u + \frac{u^2}{2} + \cdots + \frac{u^r}{r}\right).$$

Besides the triple sine and cosine functions described in the beginning, for example, these formulas give

$$\mathcal{S}_2(x) = e^x \prod_{n=1}^{\infty} \left\{ \left(\frac{1-\frac{x}{n}}{1+\frac{x}{n}}\right)^n e^{2x} \right\},$$

$$\mathcal{C}_2(x) = \prod_{n=1}^{\infty} \left\{ \left(\frac{1-\frac{x}{n-\frac{1}{2}}}{1+\frac{x}{n-\frac{1}{2}}}\right)^{n-\frac{1}{2}} e^{2x} \right\}.$$

These infinite product expressions are generalizations of

$$\mathcal{S}_1(x) = 2\pi x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2}\right) = 2\pi x \prod_{n=1}^{\infty} P_1\Big(\frac{x}{n}\Big) P_1\Big(-\frac{x}{n}\Big)$$

and

$$\mathcal{C}_1(x) = 2 \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{(n-\frac{1}{2})^2}\right) = 2 \prod_{n=1}^{\infty} P_1\left(\frac{x}{n-\frac{1}{2}}\right) P_1\left(-\frac{x}{n-\frac{1}{2}}\right).$$

We notice that the multiple sine function $\mathcal{S}_r(x)$ is a meromorphic function of order r, which is entire when r is odd. The multiple cosine function $\mathcal{C}_r(x)$ is a 2^{r-1}-multi-valued function, and

$$\tilde{\mathcal{C}}_r(x) = \mathcal{C}_r(x)^{2^{r-1}} = \prod_{n=1}^{\infty} \left\{ P_r\Big(\frac{x}{n-\frac{1}{2}}\Big) P_r\Big(-\frac{x}{n-\frac{1}{2}}\Big)^{(-1)^{r-1}} \right\}^{(2n-1)^{r-1}}$$

defines a single-valued meromorphic function of order r. In this paper we use the differential equations

$$\mathcal{S}_r'(x) = \mathcal{S}_r(x)\pi x^{r-1}\cot(\pi x)$$

and

$$\tilde{\mathcal{C}}_r'(x) = -2^{r-1}\tilde{\mathcal{C}}_r(x)\pi x^{r-1}\tan(\pi x)$$

derived from the product expressions via logarithmic differentiations.

These multiple trigonometric functions were initially constructed and investigated in [K1, K2, K3]. We refer also to Manin [M] as an excellent survey. Some details of our study have been published in [KKo, KW1, KOW] among others.

We are especially interested in the nature of the special values $\mathcal{S}_r(n/2)$ and $\mathcal{C}_r(n/2)$ for integers n. These values are intimately related to the special values of zeta functions as shown in [K1, KW1, KOW, KKo] (see also [KW4, KW5]), and furthermore these give extremal values of the functions $\mathcal{S}_r(x)$ and $\mathcal{C}_r(x)$ respectively as we determined in [KW3] when $r = 2, 3$. (See also [KW2] for its appearance in the functional equation of the higher Selberg zeta function). We know the algebraicity of $\mathcal{S}_1(1/2) = 2$ and $\mathcal{S}_2(1/2) = \sqrt{2}$, but we have no results concerning the algebraicity of $\mathcal{S}_3(1/2)$ up to now. Here we note that the fact $\mathcal{S}_2(1/2) = \sqrt{2}$ is not obvious and it is actually equivalent to Euler's famous integration ([E])

$$\int_0^{\frac{\pi}{2}} \log(\sin\phi)d\phi = -\frac{\pi}{2}\log 2$$

as shown in [KW3].

3. Multiplicative duplication formulas. We show Theorem 1.1. Since the duplication formula (A) can be proved for any degree r ($r \geq 2$) as

$$\mathcal{S}_r(2x) = \mathcal{S}_r(x)^{2^{r-1}} \mathcal{C}_r(x)^{2^{r-1}}$$

in a unified way, we show the theorem in this generalized form. In fact, observing the expressions

$$\mathcal{S}_r(2x) = e^{\frac{(2x)^{r-1}}{r-1}} \prod_{m=1}^{\infty} \left\{ P_r\Big(\frac{2x}{m}\Big) P_r\Big(-\frac{2x}{m}\Big)^{(-1)^{r-1}} \right\}^{m^{r-1}}$$

and

$$\begin{aligned}\mathcal{S}_r(x)^{2^{r-1}} &= e^{\frac{(2x)^{r-1}}{r-1}} \prod_{m=1}^{\infty} \left\{ P_r\Big(\frac{x}{m}\Big) P_r\Big(-\frac{x}{m}\Big)^{(-1)^{r-1}} \right\}^{(2m)^{r-1}} \\ &= e^{\frac{(2x)^{r-1}}{r-1}} \prod_{m=1}^{\infty} \left\{ P_r\Big(\frac{2x}{2m}\Big) P_r\Big(-\frac{2x}{2m}\Big)^{(-1)^{r-1}} \right\}^{(2m)^{r-1}},\end{aligned}$$

we see that

$$\begin{aligned}\frac{\mathcal{S}_r(2x)}{\mathcal{S}_r(x)^{2^{r-1}}} &= \prod_{m=1,\, m:\text{odd}}^{\infty} \left\{ P_r\Big(\frac{2x}{m}\Big) P_r\Big(-\frac{2x}{m}\Big)^{(-1)^{r-1}} \right\}^{m^{r-1}} \\ &= \prod_{n=1}^{\infty} \left\{ P_r\Big(\frac{x}{n-\frac{1}{2}}\Big) P_r\Big(-\frac{x}{n-\frac{1}{2}}\Big)^{(-1)^{r-1}} \right\}^{(2n-1)^{r-1}} \\ &= \mathcal{C}_r(x)^{2^{r-1}}.\end{aligned}$$

This shows (A). We notice that this property is valid for $r = 1$ also, since $\mathcal{S}_1(2x) = \mathcal{S}_1(x)\mathcal{C}_1(x)$.

Next, we prove (B). We show that

(i) both sides are 1 at $x = 0$,

(ii) logarithmic derivatives of both sides are $8\pi x^2 \cot(2\pi x)$.

If we check these facts we can conclude the equality (B). Concerning (i), we see that the left hand side is $\mathcal{S}_3(0) = 1$ and the right hand side is in fact

$$\exp\Big(\frac{7\zeta(3)}{2\pi^2}\Big)\mathcal{S}_3(0)^4 \mathcal{S}_3\Big(\frac{1}{2}\Big)^4 \mathcal{S}_2\Big(\frac{1}{2}\Big)^{-4} \mathcal{S}_1\Big(\frac{1}{2}\Big) = 1$$

which follows from

$$\mathcal{S}_3\Big(\frac{1}{2}\Big) = 2^{\frac{1}{4}} \exp\Big(-\frac{7\zeta(3)}{8\pi^2}\Big),$$

$$\mathcal{S}_2\Big(\frac{1}{2}\Big) = \sqrt{2}$$

and

$$\mathcal{S}_1\Big(\frac{1}{2}\Big) = 2.$$

To see (ii), we recall that

$$\frac{\mathcal{S}_r'}{\mathcal{S}_r}(x) = \pi x^{r-1}\cot(\pi x)$$

for $r \geq 1$ as proved in [KKo]. Hence the logarithmic derivative of the right hand side is

$$\begin{aligned}&4\frac{\mathcal{S}_3'}{\mathcal{S}_3}(x) + 4\frac{\mathcal{S}_3'}{\mathcal{S}_3}\Big(x+\frac{1}{2}\Big) - 4\frac{\mathcal{S}_2'}{\mathcal{S}_2}\Big(x+\frac{1}{2}\Big) + \frac{\mathcal{S}_1'}{\mathcal{S}_1}\Big(x+\frac{1}{2}\Big) \\ &= 4\pi x^2\cot(\pi x) + 4\pi\Big(x+\frac{1}{2}\Big)^2 \cot\pi\Big(x+\frac{1}{2}\Big) \\ &\quad - 4\pi\Big(x+\frac{1}{2}\Big)\cot\pi\Big(x+\frac{1}{2}\Big) + \pi\cot\pi\Big(x+\frac{1}{2}\Big) \\ &= 4\pi x^2\big(\cot(\pi x) - \tan(\pi x)\big) \\ &= 8\pi x^2 \cot(2\pi x),\end{aligned}$$

which is equal to the logarithmic derivative of the left hand side. This proves the theorem. □

4. Special values. We calculate several special values needed in the proofs of Theorem 1.2 and Theorem 1.3.

Proposition 4.1. *We have*

(1) $\mathcal{S}_3'(1) = -2\pi$.

(2) $\tilde{\mathcal{C}}_3'\Big(\frac{1}{2}\Big) = -2\pi\exp\Big(\frac{7\zeta(3)}{2\pi^2}\Big)$.

(3) $\tilde{\mathcal{C}}_3(1) = -16$.

(4) $\tilde{\mathcal{C}}_3'(1) = 0$.

(5) $\tilde{\mathcal{C}}_3''(1) = 64\pi^2$.

Proof. (1) Using the differential equation

$$\mathcal{S}_3'(x+1) = \mathcal{S}_3(x+1)\pi(x+1)^2\cot(\pi x)$$

and the periodicity proved in [KKo]

$$\mathcal{S}_3(x+1) = -\mathcal{S}_3(x)\mathcal{S}_2(x)^2\mathcal{S}_1(x),$$

we obtain

$$\mathcal{S}_3'(x+1) = -\pi\mathcal{S}_3(x)\mathcal{S}_2(x)^2(x+1)^2\frac{\mathcal{S}_1(x)}{\tan(\pi x)}.$$

Hence

$$\mathcal{S}_3'(1) = \lim_{x\to 0}\mathcal{S}_3'(x+1) = -\pi\mathcal{S}_3(0)\mathcal{S}_2(0)^2 \cdot 2 = -2\pi$$

since $\mathcal{S}_3(0) = \mathcal{S}_2(0) = 1$.

(2) From the duplication formula $\tilde{\mathcal{C}}_3(x) = (\mathcal{S}_3(2x))/(\mathcal{S}_3(x)^4)$ of Theorem 1.1, we calculate

$$\tilde{\mathcal{C}}_3'\Big(\frac{1}{2}\Big) = \lim_{x\to\frac{1}{2}}\frac{\tilde{\mathcal{C}}_3(x)}{x-\frac{1}{2}} = \lim_{x\to\frac{1}{2}}\frac{\mathcal{S}_3(2x)}{x-\frac{1}{2}}\cdot\mathcal{S}_3(x)^{-4}$$

$$= 2\mathcal{S}_3'(2x)\Big|_{x=\frac{1}{2}} \cdot \mathcal{S}_3\Big(\frac{1}{2}\Big)^{-4}$$
$$= 2\mathcal{S}_3'(1)\mathcal{S}_3\Big(\frac{1}{2}\Big)^{-4}$$
$$= 2\cdot(-2\pi)\cdot\Big\{2^{\frac{1}{4}}\exp\Big(-\frac{7\zeta(3)}{8\pi^2}\Big)\Big\}^{-4}$$
$$= -2\pi\exp\Big(\frac{7\zeta(3)}{2\pi^2}\Big).$$

(3) We first show the periodicity of $\tilde{\mathcal{C}}_3(x)$:

$$\tilde{\mathcal{C}}_3(x+1) = -\tilde{\mathcal{C}}_3(x)\tilde{\mathcal{C}}_2(x)^4\tilde{\mathcal{C}}_1(x)^4.$$

For this purpose we see that

$$\mathcal{S}_3(x+1) = -\mathcal{S}_3(x)\mathcal{S}_2(x)^2\mathcal{S}_1(x) \quad \text{and}$$
$$\mathcal{S}_2(x+1) = -\mathcal{S}_2(x)\mathcal{S}_1(x)$$

shown in [KKo], and we have then

$$\mathcal{S}_3(x+2) = -\mathcal{S}_3(x+1)\mathcal{S}_2(x+1)^2\mathcal{S}_1(x+1)$$
$$= -\mathcal{S}_3(x)\mathcal{S}_2(x)^4\mathcal{S}_1(x)^4.$$

Hence the periodicity

$$\tilde{\mathcal{C}}_3(x+1) = \frac{\mathcal{S}_3(2x+2)}{\mathcal{S}_3(x+1)^4} = \frac{-\mathcal{S}_3(2x)\mathcal{S}_2(2x)^4\mathcal{S}_1(2x)^4}{\mathcal{S}_3(x)^4\mathcal{S}_2(x)^8\mathcal{S}_1(x)^4}$$
$$= -\tilde{\mathcal{C}}_3(x)\tilde{\mathcal{C}}_2(x)^4\tilde{\mathcal{C}}_1(x)^4$$

follows. Letting $x=0$ we have

$$\tilde{\mathcal{C}}_3(1) = -\tilde{\mathcal{C}}_3(0)\tilde{\mathcal{C}}_2(0)^4\tilde{\mathcal{C}}_1(0)^4 = -16$$

since $\tilde{\mathcal{C}}_3(0) = \tilde{\mathcal{C}}_2(0) = 1$ and $\tilde{\mathcal{C}}_1(0) = 2$. This shows (3). We further obtain

$$\tilde{\mathcal{C}}_3'(1) = \lim_{x\to1}\tilde{\mathcal{C}}_3'(x) = \lim_{x\to1}\big(-4\pi x^2\tan(\pi x)\tilde{\mathcal{C}}_3(x)\big) = 0$$

and

$$\tilde{\mathcal{C}}_3''(1) = \lim_{x\to1}\tilde{\mathcal{C}}_3''(x)$$
$$= \lim_{x\to1}\tilde{\mathcal{C}}_3(x)\Big(16\pi^2x^4\tan^2(\pi x) - 8\pi x\tan(\pi x) - 4\pi x^2\frac{\pi}{\cos^2(\pi x)}\Big)$$
$$= -16\Big(-4\pi\cdot\frac{\pi}{1}\Big) = 64\pi^2.$$

Those show respectively the assertion (4) and (5). □

5. Duplication formulas via power series. We prove Theorem 1.2 and Theorem 1.3. Since $\mathcal{S}_3'(1)\neq 0$ and $\tilde{\mathcal{C}}_3'(1/2)\neq 0$ by (1) and (2) of Proposition 4.1, the existence of $\Phi(u)$ and $\Psi(u)$ follows immediately. Hence we calculate their first coefficients.

Proof of Theorem 1.2. From Theorem 1.1 it is sufficient to show that

$$\tilde{\mathcal{C}}_3(x) = -16 + 8\mathcal{S}_3(x)^2 + \cdots$$

around $x=1$. Setting

$$\tilde{\mathcal{C}}_3(x) = a_0 + a_1\mathcal{S}_3(x) + a_2\mathcal{S}_3(x)^2 + \cdots$$

around $x=1$, we show that $a_0=-16$, $a_1=0$ and $a_2=8$. By Proposition 4.1 (3), we have $a_0=\tilde{\mathcal{C}}_3(1) = -16$. Also, by Proposition 4.1 (4) and (1) we observe that

$$a_1 = \lim_{x\to1}\frac{\tilde{\mathcal{C}}_3(x)+16}{\mathcal{S}_3(x)} = \lim_{x\to1}\frac{\tilde{\mathcal{C}}_3'(x)}{\mathcal{S}_3'(x)} = \frac{\tilde{\mathcal{C}}_3'(1)}{\mathcal{S}_3'(1)} = 0.$$

Moreover, using this result we have

$$a_2 = \lim_{x\to1}\frac{\tilde{\mathcal{C}}_3(x)+16}{\mathcal{S}_3(x)^2} = \lim_{x\to1}\frac{\tilde{\mathcal{C}}_3'(x)}{2\mathcal{S}_3(x)\mathcal{S}_3'(x)}$$
$$= \lim_{x\to1}\frac{\tilde{\mathcal{C}}_3''(x)}{2\mathcal{S}_3'(x)^2+2\mathcal{S}_3(x)\mathcal{S}_3''(x)} = \frac{\tilde{\mathcal{C}}_3''(1)}{2\mathcal{S}_3'(1)^2} = 8$$

by Proposition 4.1 (5) and (1). This completes the proof of the theorem. □

Proof of Theorem 1.3. From the proof of Theorem 1.2 above we have seen that around $x=1$

$$\tilde{\mathcal{C}}_3(x) = -16 + 8\mathcal{S}_3(x)^2 + \cdots.$$

Now put

$$\tilde{\mathcal{C}}_3(2x) = b_0 + b_1\tilde{\mathcal{C}}_3(x) + b_2\tilde{\mathcal{C}}_3(x)^2 + \cdots$$

around $x=1/2$. Then it is obvious that $b_0 = \tilde{\mathcal{C}}_3(1) = -16$ by Proposition 4.1 (4). Next

$$b_1 = \lim_{x\to\frac{1}{2}}\frac{\tilde{\mathcal{C}}_3(2x)+16}{\tilde{\mathcal{C}}_3(x)} = \lim_{x\to\frac{1}{2}}\frac{\tilde{\mathcal{C}}_3(2x)+16}{\mathcal{S}_3(2x)}\cdot\mathcal{S}_3(x)^4$$
$$= \Big(\lim_{x\to1}\frac{\tilde{\mathcal{C}}_3(x)+16}{\mathcal{S}_3(x)}\Big)\cdot\mathcal{S}_3\Big(\frac{1}{2}\Big)^4 = \frac{\tilde{\mathcal{C}}_3'(1)}{\mathcal{S}_3'(1)}\cdot\mathcal{S}_3\Big(\frac{1}{2}\Big)^4$$
$$= 0$$

which follows from the Proposition 4.1 (4) and (1). Lastly we calculate

$$b_2 = \lim_{x\to\frac{1}{2}}\frac{\tilde{\mathcal{C}}_3(2x)+16}{\tilde{\mathcal{C}}_3(x)^2} = \lim_{x\to\frac{1}{2}}\frac{\tilde{\mathcal{C}}_3(2x)+16}{\mathcal{S}_3(2x)^2}\cdot\mathcal{S}_3(x)^8$$
$$= \Big(\lim_{x\to1}\frac{\tilde{\mathcal{C}}_3(x)+16}{\mathcal{S}_3(x)^2}\Big)\cdot\mathcal{S}_3\Big(\frac{1}{2}\Big)^8$$
$$= \frac{\tilde{\mathcal{C}}_3''(1)}{2\mathcal{S}_3'(1)^2}\cdot\mathcal{S}_3\Big(\frac{1}{2}\Big)^8 = 8\Big(2^{\frac{1}{4}}\exp\Big(-\frac{7\zeta(3)}{8\pi^2}\Big)\Big)^8$$
$$= 32\exp\Big(-\frac{7\zeta(3)}{\pi^2}\Big).$$

This proves the theorem. □

References

[E] Euler, L.: De summis serierum numeros Bernoullianos involventium. Novi commentarii academiae scientiarum Petropolitanae **14**, 129–167 (1769). [Opera Omnia I-15, pp. 91–130].

[K1] Kurokawa, N.: Multiple sine functions and Selberg zeta functions. Proc. Japan Acad., **67A**, 61–64 (1991).

[K2] Kurokawa, N.: Gamma factors and Plancherel measures. Proc. Japan Acad., **68A**, 256–260 (1992).

[K3] Kurokawa, N.: Multiple zeta functions: an example. Zeta Functions in Geometry. Adv. Stud. Pure Math., vol. 21, Kinokuniya, Tokyo, pp. 219–226 (1992).

[KKo] Kurokawa, N., and Koyama, S.: Multiple sine functions. Forum Math., **15**, 839–876 (2003).

[KOW] Kurokawa, N., Ochiai, H., and Wakayama, M.: Multiple trigonometry and zeta functions. J. Ramanujan Math. Soc., **17**, 101–113 (2002).

[KW1] Kurokawa, N., and Wakayama, M.: On $\zeta(3)$. J. Ramanujan Math. Soc., **16**, 205–214 (2001).

[KW2] Kurokawa, N., and Wakayama, M.: Higher Selberg zeta functions. (Preprint). (2002, updated in 2003).

[KW3] Kurokawa, N., and Wakayama, M.: Extremal values of double and triple trigonometric functions. (To appear in Kyushu J. Math.).

[KW4] Kurokawa, N., and Wakayama, M.: Finite ladders in multiple trigonometry. (Preprint). (2003).

[KW5] Kurokawa, N., and Wakayama, M.: Zeta regularized products for multiple trigonometric functions. (Preprint). (2003).

[M] Manin, Yu. I.: Lectures on zeta functions and motives (according to Deninger and Kurokawa). Astérisque **228**, 121–163 (1995).