On isomorphism classes of Zariski dense subgroups of semisimple algebraic groups with isomorphic p-adic closures

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(Communicated by Heisuke HIRONAKA, M. J. A., May 13, 2002)

Abstract: We prove under certain natural conditions the finiteness of the number of isomorphism classes of Zariski dense subgroups in semisimple groups with isomorphic p-adic closures.

Key words: Arithmetic subgroups; p-adic groups.

Introduction. The present paper was inspired by Mazur [Ma], where he considered various types of *local-global priciples* in number theory and also the problem, for a given number field k, to determine the *companions* of a given algebraic k-variety V (i.e. those k-forms of V, locally everywhere k_v-isomorphic to V). He also conjectured that for projective smooth varieties V over k, there are, up to k-isomorphism, only finite number of companions of V. For algebraic groups which are not necessarily linear, such a (well-known) question was answered in affirmative by Borel and Serre [BS]. We consider here an analog in the case of Zariski dense subgroups of semisimple groups. The following provides a connection with similar question. Let k be a number field, S a finite set of valuations of k, containing the set ∞ of archimedean ones. Let $\mathcal{O} = \mathcal{O}(S)$ be the ring of S-integers of k, $\mathbf{\Omega}$ be a fixed universal domain containing k. For a valuation v of k we denote by k_v the v-adic completion of k, $\mathcal{O}_v$ = v-adic integers of k_v, m_v = maximal ideal of $\mathcal{O}_v$, $\mathbf{A}$ = adèle ring of k. Algebraic groups under consideration are identified with their points over $\mathbf{\Omega}$. Assume that $G \subset \mathbf{G}(\mathcal{O})$, $\mathbf{G} \hookrightarrow \mathrm{GL}_n(\mathbf{\Omega})$, where $\mathbf{G}$ denotes the Zariski-closure of G in GL_n, $\mathbf{G}(\mathbf{B})$ will denote the $\mathbf{B}$-points of a linear algebraic group $\mathbf{G}$, with respect to the matrix realization of $\mathbf{G} \hookrightarrow \mathrm{GL}_n$ and for some ring $\mathbf{B}$. $\mathrm{Cl}_v(G)$ denotes the (v-adic) closure of G in $\mathbf{G}(k_v)$ with respect to the v-adic topology on $\mathbf{G}(k_v)$. So there attaches to a given G a collection $(\mathrm{Cl}_v(G))_v$ of v-adic closures of G, which measures how big G is locally. One may ask the following natural question:

(∗) *To what extent the collection* $(\mathrm{Cl}_v(G))_v$ *determines the group* G *up to isomorphism*? *If not, is the number of isomorphism classes finite*?

We are most interested in the finiteness aspect of above question, i.e., given topological isomorphisms $\mathrm{Cl}_v(G) \simeq \mathrm{Cl}_v(G_i)$ for all v, where i runs over a set of indices I, we ask whether the set of isomorphism classes of $\{G_i\}_i$ is finite.

These questions are closely related also to the congruence subgroup problem and strong approximation in simply connected algebraic groups in its wide sense.

It is our objective to establish the finiteness of the number of isomorphism classes in the case of Zariski-dense subgroups of almost simple simply connected groups, which are big in certain sense.

In general, this is a difficult question and we will show the finiteness to hold under certain restrictions. The first restriction is to require the groups G_i to be "big" in the sense to follow. For simplicity we restrict ourselves to the case $k = \mathbf{Q}$. Let I be a set of indices. For each $i \in I$ let G_i be a Zariski dense subgroup of simply connected absolutely almost simple $\mathbf{Q}$-group $\mathbf{G}_i \hookrightarrow \mathrm{GL}_{n_i}$, such that $G_i \subset \mathbf{G}_i(\mathbf{Z})$ and $G_i \not\simeq G_j$ if $i \neq j$. Assume that each G_i satisfies the following condition

$$\cap_p \left(\mathbf{G}_i(\mathbf{Q}) \cap \mathrm{Cl}_p(G_i)\right) = G_i. \qquad \text{(B)}$$

Here $\mathrm{Cl}_p(\cdot)$ means taking the closure in the p-adic topology of $\mathbf{G}_i(\mathbf{Q}_p)$. This condition means that G_i are "big" so that one can recover the group G_i from local closures. As it follows from Nori's Theorem [N], these groups are arithmetic subgroups in $\mathbf{G}_i(\mathbf{Z})$. A Zariski dense subgroup $G_i \subset \mathbf{G}_i$ satisfying this condition (B) such that all closures $\mathrm{Cl}_p(G_i)$ are open and compact subgroups of $\mathbf{G}_i(\mathbf{Q}_p)$ will be called *big*. Our main result can be stated as follows:

2000 Mathematics Subject Classification. Primary 22E40; Secondary 22E45.

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Theorem. *Let I be a set of indices and for $i \in I$, let G_i be a Zariski-dense subgroup of a simply connected absolutely almost simple $\mathbf{Q}$-group $\mathbf{G}_i$, such that $G_i \subset \mathbf{G}_i(\mathbf{Z})$ are mutually non-isomorphic, but all their p-adic closures are topologically isomorphic for all p and each G_i is big in $\mathbf{G}_i$. Then I is finite.*

The proof of the theorem will be given in few steps.

We fix two groups G, H from the set $\mathcal{B}(G) := \{G_i\}_{i\in I}$. By using [BS] and [Pin] we have

1. Lemma. *The set $\mathcal{B}(G)$ is a disjoint union of finitely many classes of groups G_i with $\mathbf{Q}$-isomorphic Zariski closures.*

From now on we assume that all groups G_i have $\mathbf{Q}$-isomorphic Zariski closures. We fix two groups G, H from them with Zariski closure $\mathbf{G}$, $\mathbf{H}$, respectively. Then from [Pin] and [Se] we derive

2. Lemma. *For all p, isomorphisms $f_p : \mathrm{Cl}_p(G) \simeq \mathrm{Cl}_p(H)$ give rise to $\mathbf{Q}_p$-isomorphisms $\bar{f}_p : \mathbf{G} \to \mathbf{H}$ where for almost all p, $\bar{f}_p$ is a $\mathbf{Z}_p$-polynomial isomorphism with respect to the given matrix realization of the groups $\mathbf{G}$ and $\mathbf{H}$.*

We need the following lemma in the sequel in order to realise $\mathrm{Aut}(\mathbf{G})$ as linear algebraic group over $\mathbf{Q}$.

3. Lemma. *With above notation, let $f_1, \dots, f_N$ be $\mathbf{Q}$-rational functions over $\mathbf{G}$ which are linearly independent over $\mathbf{Q}$. Then there exists $x_1, \dots, x_N \in \mathbf{G}(\mathbf{Q})$ such that*

$$\det(f_i(x_j))_{1\le i,j,\le N} \in \mathbf{Q} \setminus \{0\}.$$

Denote by $M = \mathrm{Aut}(\mathbf{G})$ the group of rational automorphisms of $\mathbf{G}$. It is well-known that M has a natural structure of linear $\mathbf{Q}$-algebraic group (see, e.g., [BS, HM]) We need a specific realization of the group M, which plays a crucial role in our proof, as follows. Recall that $\mathbf{A}$ denotes the adèle ring of $\mathbf{Q}$.

4. Proposition. *With above notation there is a realization of M as a linear algebraic $\mathbf{Q}$-group such that for every $H \in \mathcal{B}(G)$ and for any $\mathbf{Q}$-isomorphism $g : \mathbf{H} \to \mathbf{G}$, the family $(g \circ f_p)$ (p runs over all prime numbers) is belong to $M(\mathbf{A})$.*

Denote by

$$\mathcal{C}(G) = \Big\{(f_p) \in \prod_p M(\mathbf{Q}_p) : f_p(\mathrm{Cl}_p(G)) = \mathrm{Cl}_p(G),\ \forall p, \text{ and } f_p \in M(\mathbf{Z}_p) \text{ for almost all } p\Big\}.$$

It is clear that $\mathcal{C}(G)$ is an infinite subgroup of $M(\mathbf{A})$. Next we want to parametrize the set $\mathcal{B}(G)$ by assigning to each $H \in \mathcal{B}(G)$ a double coset class in $M(\mathbf{Q})\backslash M(\mathbf{A})/\mathcal{C}(G)$ defined as follows:

If $g : \mathbf{H} \simeq \mathbf{G}$ is a $\mathbf{Q}$-isomorphism, $\bar{f}_p : \mathbf{G} \simeq \mathbf{H}$ is the isomorphism extending $f_p : \mathrm{Cl}_p(G) \simeq \mathrm{Cl}_p(H)$ for all p, then we set

$$a(G, H) := M(\mathbf{Q})(g \circ \bar{f}_p)\mathcal{C}(G).$$

According to Proposition 4, $(g \circ \bar{f}_p) \in M(\mathbf{A})$ so $a(G, H)$ is an element of the set of double coset classes $M(\mathbf{Q})\backslash M(\mathbf{A})/\mathcal{C}(G)$.

5. Proposition. *The correspondence defined above is a well-defined map.*

The injectivity of the map $H \mapsto a(G, H)$ now follows from the following

6. Proposition. *If (G, H) and (G, K) have the same double coset class then $H = K$.*

Preceding observations show that the cardinality of $\mathcal{B}(G)$ is not greater than the cardinality of $M(\mathbf{Q})\backslash M(\mathbf{A})/\mathcal{C}(G)$. We want to show that the latter is finite. Define

$$\mathcal{D} = \mathcal{D}(G) := \{(a_p) \in \mathcal{C}(G) : a_p \in M(\mathbf{Z}_p),\ \forall p\},$$

i.e., $\mathcal{D} = \mathcal{C}(G) \cap M(\mathbf{A}(\infty))$, where $\mathbf{A}(\infty)$ denotes the subring of finite adèles of $\mathbf{A}$. In particular we have

$$\mathrm{Card}(M(\mathbf{Q})\backslash M(\mathbf{A})/\mathcal{C}(G)) \le \mathrm{Card}(M(\mathbf{Q})\backslash M(\mathbf{A})/\mathcal{D}.$$

The following proposition plays a crucial role in the proof of the finiteness of $\mathrm{Card}(M(\mathbf{Q})\backslash M(\mathbf{A})/\mathcal{D})$.

7. Proposition. *There is only a finite number of subgroups of a given finite index m in $\mathbf{G}(\mathbf{Z}_p)$.*

8. Remarks. We can use similar arguments as in the proof of Proposition 7 to prove (compare also with [Seg]) that for a given compact p-adic analytic group, the number of its subgroups of given index m is finite. By using this, in combination with Bruhat - Tits [BrT] result about maximal compact subgroups of reductive p-adic groups, one can show that there is only a finite number of subgroups of $\mathbf{G}(\mathbf{Q}_p)$ containing $\mathbf{G}(\mathbf{Z}_p)$ with given index m, up to $\mathbf{G}(\mathbf{Q}_p)$-conjugacy.

Now we denote by

$$M(\mathbf{Z}_p, \mathrm{Cl}_p(G)) := \{f \in M(\mathbf{Z}_p) : f(\mathrm{Cl}_p(G)) = \mathrm{Cl}_p(G)\}.$$

From [MVW], Theorem 7.3, or [No], Theorem 5.4, we know that $\mathrm{Cl}_p(G) = \mathbf{G}(\mathbf{Z}_p)$ for almost all p (say, for all p outside a finite set W of primes). By the choice of the functions f_j (in the proof of Proposition 4), they are $\mathbf{Z}$-polynomial functions. So if $f \in M(\mathbf{Z}_p)$ then we have $f(\mathbf{G}(\mathbf{Z}_p)) = \mathbf{G}(\mathbf{Z}_p)$. Hence for $p \notin W$ we have

$$M(\mathbf{Z}_p, \mathrm{Cl}_p(G)) = M(\mathbf{Z}_p).$$

We need also the following

9. Proposition. *$M(\mathbf{Z}_p, \mathrm{Cl}_p(G))$ is of finite index in $M(\mathbf{Z}_p)$.*

10. Proposition. *With above notation we have*

$$\mathrm{Card}(M(\mathbf{Q})\backslash M(\mathbf{A})/\mathcal{D}) < \infty.$$

Now the proof of the Theorem follows from main results of [Bor] and these results.

Acknowledgements. I would like to thank Professor B. Mazur for his interest and valuabale suggestions regarding this paper, Professor R. Pink for sending his papers which help a great deal the work over this paper. This work has been done thanks to the support by NCST, The Abdus Salam I.C.T.P. and Swedish International Developement Agency (S.I.D.A.).

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