A note on the mean value of the zeta and L-functions. XI

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(Communicated by Shokichi IYANAGA, M. J. A., Jan. 15, 2002)

Abstract: The present note reports an optimal bound for a version of the spectral fourth power moment of Hecke L-functions associated with Maass forms over the full modular group, in which the spectral parameter runs over short intervals. Consequentially, a new hybrid subconvexity bound is attained for individual values of those L-functions on the critical line.

Key words: Hybrid subconvexity bound; Hecke L-function; Maass form; Bruggeman–Kuznetsov sum formula; binary additive divisor sum.

1. Introduction. Let $\Gamma = \mathrm{PSL}_2(\mathbf{Z})$, and $\mathbf{H}$ the hyperbolic upper half plane. The cuspidal subspace of $L^2(\Gamma\backslash\mathbf{H})$ has a maximal orthonormal system composed of Maass forms ψ_j, $j = 1, 2, \ldots$, such that ψ_j corresponds to the eigenvalue $1/4 + \kappa_j^2$ of the hyperbolic Laplacian, with $0 < \kappa_1 \le \kappa_2 \le \cdots$. We may assume that ψ_j are simultaneous eigenfunctions of all Hecke operators with corresponding eigenvalues $t_j(n)$, $\mathbf{Z} \ni n > 0$, and that $\psi_j(-\overline{z}) = \epsilon_j \psi_j(z)$ with $\epsilon_j = \pm 1$. Then, the Hecke L-function attached to ψ_j is defined by

$$H_j(s) = \sum_{n=1}^{\infty} t_j(n) n^{-s}, \quad \operatorname{Re} s > 1.$$

This continues to an entire function, and satisfies the functional equation

$$H_j(s) = \chi_j(s) H_j(1-s), \tag{1.1}$$

with

$$\chi_j(s) = 2^{2s-1}\pi^{2(s-1)}\Gamma(1-s+i\kappa_j)\Gamma(1-s-i\kappa_j) \cdot \{\epsilon_j \cosh \pi\kappa_j - \cos \pi s\}.$$

For details of the above, we refer to [Mo2].

Comparing (1.1) with the functional equation for the square of the Riemann zeta-function, one may conjecture the hybrid subconvexity bound

$$H_j\left(\frac{1}{2} + it\right) \ll (t + \kappa_j)^{1/3+\varepsilon}, \quad t \ge 0. \tag{1.2}$$

Here and in what follows, ε is an arbitrary small positive constant, and all implied constants may depend on it. For a fixed κ_j and growing t, a result due to T. Meurman [Me] implies (1.2). On the other hand, for a fixed t and growing κ_j, the estimate (1.2) was proved conditionally by H. Iwaniec [Iw]; and he pointed out later to the first named author that his argument gives, still for t fixed, the bound $\ll \kappa_j^{5/12+\varepsilon}$ unconditionally. Further, as a remarkable breakthrough, A. Ivić [Iv] succeeded in proving (1.2) for $t = 0$ by a method quite different from those previously applied (see [Mo3] for the announcement). His starting point was an arithmetical expression [Mo2, Lemma 3.8], due to the second named author, for the weighted spectral mean square of $H_j(1/2)$. Ivić's bound for $H_j(1/2)$ follows as a corollary of his result

$$\sum_{|\kappa_j - K| \le 1} \alpha_j H_j^3\left(\frac{1}{2}\right) \ll K^{1+\varepsilon}, \quad K \ge 1,$$

and the well-known inequalities $H_j(1/2) \ge 0$, $\alpha_j \gg \kappa_j^{-\varepsilon}$. Here $\alpha_j = |\varrho_j|^2/\cosh \pi\kappa_j$ with ϱ_j the first Maass–Fourier coefficient of ψ_j. Motivated by this advance with the cubic moment, the first named author [Ju2] turned to the fourth moment, establishing the mean value estimate

$$\sum_{|\kappa_j - K| \le K^{1/3}} \alpha_j H_j^4\left(\frac{1}{2}\right) \ll K^{4/3+\varepsilon}. \tag{1.3}$$

The identity [Mo2, Lemma 3.8] played again a crucial rôle in his proof. Also, as a new basic tool, a use was made of an explicit formula, due to the second named author [Mo1], for the binary additive divisor sum

$$D(f; W) = \sum_{n=1}^{\infty} d(n) d(n+f) W\left(\frac{n}{f}\right), \tag{1.4}$$

2001 Mathematics Subject Classification. Primary 11F66; Secondary 11F72.

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where d is the divisor function, $f \geq 1$, and W an arbitrary smooth function with a compact support on the positive real axis. It should be observed that (1.3) reproves Ivić's bound for $H_j(1/2)$ without the non-negativity of those central values.

Now, as a deeper evidence supporting our conjecture (1.2), we report

Theorem. *Assume that K is large, and*

$$(1.5) \qquad 0 \leq t \leq K^{1-\theta}, \quad \frac{1}{3} < \theta < 1.$$

Then it holds that

$$(1.6) \qquad \sum_{|\kappa_j - K| \leq K^{1/3}} \alpha_j \left| H_j\left(\frac{1}{2} + it\right)\right|^4 \ll K^{4/3+\varepsilon}.$$

The implicit constant depends only on ε and θ.

Hence, (1.2) is valid in the range $0 \leq t \leq \kappa_j^{2/3-\varepsilon}$, uniformly in H_j. The proof that we shall sketch below is, in principle, an elaboration of [Ju2]. Thus, the explicit formula for $D(f;W)$ is utilized in much the same context as in [Ju2]. However, there is a significant change of argument as well: the Voronoï formula is used instead of [Mo2, Lemma 3.8]. A detailed proof is available in [JM], which is intended for publication. Meurman's argument actually gives (1.2) for $t \geq \kappa_j^3$, and thus it remains to consider the range $\kappa_j^{2/3-\varepsilon} \leq t \leq \kappa_j^3$. To this issue, we shall return elsewhere.

2. Reduction. We shall indicate salient points of our proof of (1.6). In this section, a reduction to a binary additive divisor sum will be performed.

Lemma 1. *Let K be a sufficiently large parameter, and assume that $K^\varepsilon \leq G \leq K^{1-\varepsilon}$, $0 \leq t \leq (1/2)K$, $N = (K^2 - t^2)/(4\pi^2)$. Further, let $\lambda(x)$ be any smooth function supported compactly on the positive real axis, satisfying*

$$\lambda(x) = \lambda\left(\frac{1}{x}\right), \quad \int_0^\infty \lambda(\xi)\frac{d\xi}{\xi} = 1.$$

Put

$$(2.1) \qquad I_j = \sum_{n=1}^\infty a_j(n) n^{-1/2-it} \Lambda\left(\frac{n}{N}\right),$$

where $a_j(n) = \sum_{l^2|n} d(n/l^2) t_j(n/l^2)$ and

$$\Lambda(x) = \int_x^\infty \lambda(\xi)\frac{d\xi}{\xi}, \quad x > 0.$$

With this, we define E_j by

$$H_j^2\left(\frac{1}{2} + it\right) = I_j + \chi_j^2\left(\frac{1}{2} + it\right)\overline{I_j} + E_j,$$

where χ_j is as in (1.1). Then we have

$$(2.2) \qquad \sum_{|\kappa_j - K| \leq G} \alpha_j |E_j|^2 \ll (G^2 + K^{2/3} + t)K^\varepsilon.$$

The implicit constant depends only on ε and λ.

Proof. Ramachandra's basic argument and Iwaniec's spectral large sieve (see Lemma 3.9 and Theorem 3.3 of [Mo2], respectively) suffice to prove this. Details are given in [JM]. For the proof of our theorem, this version of the approximate functional equation for Hecke series is, actually, somewhat redundant. However, the assertion (2.2) appears to have an independent interest. □

Let I_j be as above, and put

$$h(r) = K^{-2}\left(r^2 + \frac{1}{4}\right)\left[\exp\left(-\left(\frac{r-K}{G}\right)^2\right) + \exp\left(-\left(\frac{r+K}{G}\right)^2\right)\right],$$

where $G = K^\theta$ with θ as in (1.5). By virtue of (2.2), it is enough to show that

$$(2.3) \qquad \mathcal{S} = \sum_{j=1}^\infty \alpha_j |I_j|^2 h(\kappa_j) \ll GK^{1+\varepsilon}.$$

To achieve this, we shall modify $\mathcal{S}$ in several steps. The most significant contributions will be denoted by $\mathcal{S}_1$, $\mathcal{S}_2$, ...; accordingly the proof of (2.3) is reduced to the same for the $\mathcal{S}_\nu$. To this end, we shall apply approximation arguments such as the saddle-point method. For a typical expression Z to be dealt with in this context, we shall employ the notation $Z \sim Z_0$ to indicate the following procedure: we shall have an approximation $Z = Z_0 + Z_1 + O(X)$, where Z_0 is the leading term, and the lesser term Z_1 oscillates in the same mode as Z_0, while X contributes negligibly to the relevant $\mathcal{S}_\nu$. Then, the replacement of Z by Z_0 should cause no differences in bounding $\mathcal{S}_\nu$.

By the definition (2.1), the estimation of $\mathcal{S}$ is reduced to that of

$$\mathcal{S}_1 = \sum_{m=1}^\infty \sum_{n=1}^\infty \psi(m)\psi(n) d(m) d(n) (mn)^{-1/2}\left(\frac{m}{n}\right)^{it} \cdot \sum_{j=1}^\infty \alpha_j h(\kappa_j) t_j(m) t_j(n),$$

with ψ being a smooth weight function supported in an interval $[M, 2M]$, $M \ll K^2$. We apply the

Bruggeman–Kuznetsov sum formula [Mo2, Theorem 2.2] to the last sum over j with the present choice of the weight h. The contribution to $\mathcal{S}_1$ of the continuous spectrum can be ignored, since it is non-positive; and the delta-term contributes by $O(GK(\log K)^4)$. Thus we need to consider closely the Kloosterman part only. It is equal to

$$\sum_{\ell=1}^{\infty} \frac{1}{\ell} S(m,n;\ell) \widetilde{h}\left(\frac{4\pi\sqrt{mn}}{\ell}\right),$$

where $S(m,n;\ell)$ is a Kloosterman sum mod ℓ, and

$$\widetilde{h}(x) = \frac{2i}{\pi}\int_{-\infty}^{\infty} \frac{rh(r)J_{2ir}(x)}{\cosh(\pi r)}dr. \tag{2.4}$$

The sum can be truncated to $1 \le \ell \le K^A$ for some constant A, which is a result of the shift of the contour in (2.4) to $\operatorname{Im} r = -1$. In the rest of the sum, the relevant values of $x = 4\pi\sqrt{mn}/\ell$ are actually large, namely $x \ge GK(\log K)^{-1}$. To see this, insert into (2.4) the expression

$$J_{2ir}(x) = \frac{2}{\pi}\int_0^{\infty} \cos(2ru) \cdot \sin(x\cosh u - ir\pi)du \tag{2.5}$$

(the formula (12) on p. 180 of [Wa]), and apply integration by parts repeatedly.

In other words, we may impose the condition

$$1 \le \ell \ll \ell_0 = M(GK)^{-1}\log K. \tag{2.6}$$

On this, we shall further consider (2.4). Transform the integral to the one over the positive real axis; and restrict the integration to $|r-K| \le G\log K$. Then, observe that the saddle point method applied to (2.5) gives

$$J_{2ir}(x) \sim \frac{1}{\pi\sqrt{2x}} \exp\left(i\omega(r,x) + \pi r - \frac{1}{4}\pi i\right), \tag{2.7}$$

where $\omega(r,x) = x(1-2(r/x)^2)$, and the condition $\theta > 1/3$ is essential. This reduces the estimation of $\mathcal{S}_1$ to that of

$$\mathcal{S}_2 = GK\sum_{m=1}^{\infty}\frac{\psi(m)d(m)}{m^{3/4-it}} \sum_{1\le\ell\ll\ell_0}\frac{1}{\sqrt{\ell}}\sum_a e\left(\frac{am}{\ell}\right) \cdot \sum_{n=1}^{\infty}\frac{\psi(n)d(n)}{n^{3/4+it}} e\left(\frac{\tilde{a}n}{\ell}\right)\exp\left(\delta_1 i\omega(r,x)\right),$$

where $e(x) = \exp(2\pi i x)$, $\delta_1 = \pm 1$, $x = 4\pi\sqrt{mn}/\ell$, $|r-K| \le G\log K$, $a \bmod \ell$ with $(a,\ell)=1$, and $a\tilde{a} \equiv 1 \bmod \ell$.

Next, the last sum over n is transformed with the Voronoï formula (see, e.g., [Ju1]). In the resulting expression, the leading term and the part of the new infinite sum that involves the K-Bessel function are readily seen to be negligible, because of (2.6). Hence we are left with

$$-\frac{2\pi}{\ell}\sum_{n=1}^{\infty} d(n)e\left(-\frac{na}{\ell}\right)\int_0^{\infty} Y_0\left(\frac{4\pi\sqrt{ny}}{\ell}\right)g(y)dy,$$

where $g(y) = \psi(y)y^{-3/4-it}\exp(\delta_1 i\omega(r, 4\pi\sqrt{my}/\ell))$, and Y_0 is a Bessel function in the notation of [Wa]. Note that $Y_0(x) \sim (2/(\pi x))^{1/2}\sin(x-(1/4)\pi)$ (see the formula (4) on p. 199 of [Wa]). Thus, in place of $\mathcal{S}_2$, we shall treat the sum

$$\mathcal{S}_3 = GK\sum_{m=1}^{\infty}\frac{\psi(m)d(m)}{m^{3/4-it}}\sum_{1\le\ell\ll\ell_0}\frac{1}{\ell} \cdot \sum_{n=1}^{\infty}\frac{d(n)}{n^{1/4}}c_\ell(n-m)I(\ell,m,n;\delta_1,\delta_2),$$

where c_ℓ is the Ramanujan sum mod ℓ, and

$$I(\ell,m,n;\delta_1,\delta_2) = \int_0^{\infty}\frac{\psi(y)}{y^{1+it}} \cdot \exp\left(\delta_1 i\omega\left(r,\frac{4\pi\sqrt{my}}{\ell}\right) + \frac{4\pi\delta_2 i\sqrt{ny}}{\ell}\right)dy$$

with $\delta_2 = \pm 1$.

A multiple application of integration by parts shows that I is negligibly small, provided either $\delta_1 = \delta_2$ or $n - m \gg \left(t\ell + (K\ell)^2/M\right)K^\varepsilon$. Thus, in view of (1.5) and (2.6), we may restrict ourselves to the situation where $\delta_1 = -\delta_2$ and $|m-n| \le G^{-2}MK^\varepsilon$. With this, if $m = n$, then $\delta_1 = 1$ can be assumed, since otherwise integration by parts shows that I is negligibly small. On the other hand, if $m = n$ and $\delta_1 = 1$, then $I \sim 2e^{-(1/4)\pi i}(\pi/t)^{1/2}y_0^{-it}\psi(y_0)$, where y_0 is the saddle point; and the harmless assumption $t \ge K^\varepsilon$ has been introduced. That is, the estimation of the diagonal part of $\mathcal{S}_3$ is reduced to that of

$$\frac{GK}{\sqrt{t}}\sum_{m=1}^{\infty}\frac{\psi(m)d^2(m)}{m}\sum_{\ell=1}^{\infty}\frac{\varphi(\ell)}{\ell^{1+2it}}\psi\left(\frac{r^4\ell^2}{16\pi^2 mt^2}\right),$$

where φ is the Euler function, and the condition $\ell \ll \ell_0$ has been eliminated after a simple observation on the size of y_0. The inner sum can be expressed as an integral using the Riemann zeta-function and the Mellin transform of ψ. Then, invoking the upper bound for the zeta-function on the imaginary axis together with its lower bound on the line with the real part equal to 1, we find that the last expression is $\ll GK(\log K)^6$.

Hence, anticipating that the case $n-m<0$ is analogous to the case $n-m>0$, the estimation of $\mathcal{S}_3$ is reduced to that of

$$GK\sum_{1\le f\ll f_0}\sum_{1\le \ell\ll \ell_0}\frac{c_\ell(f)}{\ell}\sum_{m=1}^{\infty}\frac{\psi(m)d(m)d(m+f)}{m^{3/4-it}(m+f)^{1/4}}\cdot I(\ell,m,m+f;\delta_1,-\delta_1)$$

with $f_0=G^{-2}MK^{\varepsilon}$. The change of variable $v=4\pi\ell^{-1}\sqrt{fy}(u+1/2)^{-1/2}$ in the integral I transforms this into

$$2(16\pi)^{it}GK\sum_{1\le f\ll f_0}\frac{1}{f^{1-2it}}\sum_{1\le \ell\ll \ell_0}\frac{c_\ell(f)}{\ell^{1+2it}}\cdot\sum_{m=1}^{\infty}d(m)d(m+f)U\left(\frac{m}{f}\right),$$

where

$$U(u)=\psi(fu)u^{-3/4}(u+1)^{-1/4}\cdot\left(1+\frac{1}{2u}\right)^{-it}J(f,\ell,u)$$

with

$$J(f,\ell,u)=\int_0^\infty\psi\left(\frac{(\ell v)^2}{16\pi^2 f}\left(u+\frac12\right)\right)v^{-1-2it}\cdot\exp\left(-\delta_1 i\frac{v\sqrt{u+1/2}}{\sqrt{u}+\sqrt{u+1}}-2\delta_1 i\frac{r^2}{v\sqrt{u(u+1/2)}}\right)dv.$$

The condition (1.5) and $f\ll f_0$ imply that $U(u)\sim u^{-1}\psi(fu)J(f,\ell,u)$, and that the last exponentiated factor is $\sim\exp(-(1/2)\delta_1 iv-2\delta_1 ir^2/(uv))$. Here the condition $\theta>1/3$ is essential, as in (2.7). We are thus left with

$$\mathcal{S}_4=GK\sum_{1\le f\ll f_0}\frac{1}{f^{1-2it}}\sum_{1\le \ell\ll \ell_0}\frac{c_\ell(f)}{\ell^{1+2it}}\cdot\sum_{m=1}^{\infty}d(m)d(m+f)V\left(\frac{m}{f}\right),\tag{2.8}$$

where

$$V(u)=u^{-1}\int_0^\infty\xi(f,\ell,u,v)\cdot\exp\left(-\frac12\delta_1 iv-2\delta_1 i\frac{r^2}{uv}\right)v^{-1-2it}dv,\tag{2.9}$$

with

$$\xi(\tau_1,\tau_2,u,v)=\psi(\tau_1 u)\psi\left(\frac{(\tau_2 v)^2(u+1/2)}{16\pi^2\tau_1}\right).$$

We note that the case $n-m<0$ mentioned above corresponds to the situation where the sign of $-(1/2)\delta_1 iv$ in (2.9) is altered, and the factor $\psi(\tau_1 u)$ in the last expression replaced by $\psi(\tau_1(u+1))$.

3. Spectral argument. To the inner-most sum of (2.8) we shall apply the explicit spectral decomposition of $D(f;W)$ defined by (1.4):

Lemma 2. *Put*

$$\begin{aligned}Y_f(u)&=(\log u)\log(u+1)+4\frac{\sigma''}{\sigma}(f)\\&\quad+\left\{a_1-\log f+2\frac{\sigma'}{\sigma}(f)\right\}\log(u(u+1))\\&\quad+(a_1-\log f)^2+a_2+4\frac{\sigma'}{\sigma}(f)(a_1-\log f),\end{aligned}$$

where $\sigma^{(\nu)}(f)=\sum_{d|f}d(\log d)^\nu$, *and* a_ν *are absolute constants. Then we have*

$$\begin{aligned}D(f;W)&=\frac{6}{\pi^2}\sigma(f)\int_0^\infty Y_f(u)W(u)du\\&\quad+f^{1/2}\sum_{j=1}^{\infty}\alpha_j t_j(f)H_j^2\left(\frac12\right)\Theta(\kappa_j;W)\\&\quad+D^{(c)}(f;W)+D^{(h)}(f;W).\end{aligned}\tag{3.1}$$

Here $D^{(c)}$ *and* $D^{(h)}$ *stand for the contributions of the continuous spectrum and the holomorphic cusp forms, respectively*; *and*

$$\Theta(\kappa;W)=\frac12\int_0^\infty\mathrm{Re}\left\{\left(1+\frac{i}{\sinh\pi\kappa}\right)\frac{\Gamma^2(1/2+i\kappa)}{\Gamma(1+2i\kappa)}\cdot u^{-1/2-i\kappa}F\left(\frac12+i\kappa,\frac12+i\kappa;1+2i\kappa;-\frac1u\right)\right\}W(u)du,$$

with the hypergeometric function F.

Proof. This is a minor modification of [Mo2, Theorem 3]. There it is implicitly assumed that the weight function W is real-valued. In the above, W can take complex values as well. The constructions of $D^{(c)}$ and $D^{(h)}$ are analogous to the contribution of the discrete spectrum. □

Of the terms on the right side of (3.1), the contribution of the second term is the most significant, and will be treated below. The term $D^{(c)}$ is analogous but only easier, and $D^{(h)}$ is negligibly small, as usual with the contribution of holomorphic cusp forms. Dealing with the leading term, we perform the change of variable $u\mapsto w/v$ in the relevant double integral (cf. (3.4) below), and we come to a situation similar to that of the diagonal part of $\mathcal{S}_3$, though more complicated.

Now, the function $\Theta(\kappa;V)$ is approximated by using the fact that the hypergeometric function there is $\sim 2^{1+2i\kappa}(1+\sqrt{1+1/u})^{-1-2i\kappa}$, which follows via a quadratic transformation. Hence the estimation of $\mathcal{S}_4$ is reduced to that of

$$(3.2)\ \mathcal{S}_5 = GK \sum_{1\le f\ll f_0} \frac{1}{f^{1/2-2it}} \sum_{1\le \ell\ll \ell_0} \frac{c_\ell(f)}{\ell^{1+2it}} \cdot \sum_{j=1}^{\infty} 2^{2\delta_3 i\kappa_j} \alpha_j t_j(f) H_j^2\left(\frac12\right)\left(1+\delta_3 \frac{i}{\sinh \pi\kappa_j}\right) \cdot \frac{\Gamma^2(1/2+\delta_3 i\kappa_j)}{\Gamma(1+2\delta_3 i\kappa_j)} \Xi(f,\ell,\kappa_j,\delta_1,\delta_3),$$

where $\delta_3 = \pm 1$, and $\Xi(\tau_1,\tau_2,\kappa,\delta_1,\delta_3)$ is

$$(3.3)\ \int_0^\infty\int_0^\infty \xi(\tau_1,\tau_2,u,v)\exp\left(-\frac12\delta_1 iv - 2\delta_1 i\frac{r^2}{uv}\right) \cdot (\sqrt{u}+\sqrt{u+1})^{-1-2\delta_3 i\kappa} v^{-2it}\frac{dudv}{uv},$$

with ξ as above.

We are going to estimate Ξ in (3.2). Let us consider first the case $\delta_1 = -\delta_3$. Integrating repeatedly by parts with respect to u in (3.3), we see that we may truncate (3.2) so that ℓ, $\kappa_j \le K^\varepsilon$. Then, in (3.3) we use the fact $(\sqrt{u}+\sqrt{u+1})^{i\kappa} \sim (2\sqrt{u})^{i\kappa}$ and the change of variable $u \mapsto w/v$ so that we consider, instead,

$$(3.4)\ \int_0^\infty\int_0^\infty \xi\left(\tau_1,\tau_2,\frac{w}{v},v\right)\exp\left(-\frac12\delta_1 iv - 2\delta_1 i\frac{r^2}{w}\right) \cdot v^{-1/2-\delta_1 i\kappa-2it} w^{-3/2+\delta_1 i\kappa} dvdw.$$

We may obviously assume further that $f \ge K^\varepsilon$; and the presence of the factor ξ allows us to introduce the truncation $v \ge 1$. Then the saddle point method yields that the v-integral is bounded uniformly, and (3.4) is $O(K^\varepsilon/\sqrt{M})$. Invoking the uniform bound $t_j(f) \ll f^{1/4+\varepsilon}$, we see that if $\delta_3 = -\delta_1$, then $\mathcal{S}_5 \ll G^{-1/2}K^{3/2+c\varepsilon}$ with a certain $c > 0$. This is negligible.

Hence, let us assume that $\delta_1 = \delta_3$. The saddle point u_0 of the u-integral in (3.3) satisfies $\kappa^2 v^2 u_0^3 = 4r^4(u_0+1)$. This is relevant only if $\kappa \asymp K^2\ell/M \ll G^{-1}K\log K$ because of the factor ξ and (2.6). In particular, we may truncate the inner-most sum in (3.2) so that $\kappa_j \le Q_0 = G^{-1}K^{1+\varepsilon}$. Thus the estimation of $\mathcal{S}_5$ is reduced to that of the expression

$$GK\log K \max_{1\le Q\le Q_0} Q^{-1/2} \sum_{Q\le\kappa_j\le 2Q} \alpha_j H_j^2\left(\frac12\right) \cdot \left|\sum_{1\le f\ll f_0} \frac{t_j(f)}{f^{1/2-2it}} \sum_{1\le\ell\ll\ell_0} \frac{c_\ell(f)}{\ell^{1+2it}} \Xi(f,\ell,\kappa_j,\delta_1,\delta_1)\right|.$$

We are going to apply the spectral large sieve and the spectral fourth moment of $H_j(1/2)$ (Theorems 3.3 and 3.4 of [Mo2], respectively). To this end, we separate the parameters f, ℓ trapped in Ξ, using the Mellin transform $\Xi^*(s_1,s_2,\kappa)$ of $\Xi(\tau_1,\tau_2,\kappa,\delta_1,\delta_1)$ with respect to τ_1 and τ_2. Our task is then to estimate Ξ^*. Since Ξ is smooth in τ_1 and τ_2, that is essentially the same as to estimate Ξ itself. To do the latter, we note that the condition $\kappa \ll Q_0$ implies that $(\sqrt{u}+\sqrt{u+1})^{i\kappa} \sim (2\sqrt{u})^{i\kappa}$. Hence, the u-integral is, via the saddle-point method,

$$\sim c(r,\kappa)\xi\left(\tau_1,\tau_2,2\frac{r^2}{\kappa v},v\right) v^{1/2+\delta_1 i\kappa} r^{-1},$$

where $|c(r,\kappa)| = (\pi/2)^{1/2}$, and we have imposed the harmless assumption $K^\varepsilon \le \kappa$. We use this approximation and integrate further with respect to v. We then find that $\Xi \ll K^{-1}$. In other words, we have $(\partial/\partial\tau_1)^2(\partial/\partial\tau_2)^2\Xi \ll (\tau_1\tau_2)^{-2}K^{-1}$ as well. Hence, we get, for $\operatorname{Re} s_1$, $\operatorname{Re} s_2 = (\log K)^{-1}$,

$$\Xi^*(s_1,s_2,\kappa) \ll |s_1 s_2|^{-2}K^{-1}(\log K)^2,$$

which leads us again to the negligible bound $\mathcal{S}_5 \ll G^{-1/2}K^{3/2+\varepsilon}$.

Finally, observing that the change in (2.9) mentioned at the end of the previous section does not require any alternations in the above argument, we end the proof of our theorem.

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