A characterization of the second Veronese embedding into a complex projective space

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Abstract: We study curves of order 2 from the viewpoint of submanifold theory. We give a characterization of the parallel Kähler embeddings of a complex projective space into an ambient complex projective space from this point of view. This characterization is an improvement of the results in [N, PS].

Key words: Veronese embedding; curves of order 2; complex projective spaces.

1. Introduction. When we study Riemannian submanifolds, it is natural to investigate their properties by observing the extrinsic shape of their geodesics. In this paper we pay attention to the extrinsic shape of geodesics on a complex projective space $\mathbf{C}P^n(c)$ of constant holomorphic sectional curvature c in a complex projective space $\mathbf{C}P^N(\tilde{c})$ of constant holomorphic sectional curvature $\tilde{c}$ through a Kähler isometric full immersion. By virtue of the classification theorem ([C, NO]) this Kähler immersion is nothing but a Kähler embedding $f_k : \mathbf{C}P^n(c/k) \to \mathbf{C}P^N(c)$ given by

$$[z_i]_{0\leqq i\leqq n} \mapsto \left[\sqrt{\frac{k!}{k_0!\cdots k_n!}}z_0^{k_0}\cdots z_n^{k_n}\right]_{k_0+\cdots+k_n=k},$$

where $[*]$ means the point of the projective space with the homogeneous coordinates $*$ and $N = (n+k)!/(n!k!) - 1$. We usually call f_k the k-th *Veronese embedding*. It is known that the second fundamental form of f_k is parallel if and only if $k = 1$ or $k = 2$.

These parallel embeddings f_k $(k = 1, 2)$ have various geometric properties. For example, the second Veronese embedding f_2 maps each geodesic on the submanifold $\mathbf{C}P^n(c/2)$ to a circle of curvature $\sqrt{c}/2$ in a real projective plane $\mathbf{R}P^2(c/4)$ of curvature $c/4$ which is a totally real totally geodesic submanifold of the ambient manifold $\mathbf{C}P^{n(n+3)/2}(c)$. Using such a property, for a Kähler isometric full immersion $f : M_n \to \mathbf{C}P^N(c)$ of an n-dimensional Kähler manifold into an N-dimensional complex projective space of constant holomorphic sectional curvature c, K. Nomizu [N] showed that either $M_n = \mathbf{C}P^n(c)$ and $N = n$, or $M_n = \mathbf{C}P^n(c/2)$, $N = n(n+3)/2$ and f is locally equivalent to the second Veronese embedding f_2 if and only if for each geodesic γ on M_n the curve $f \circ \gamma$ is a circle on $\mathbf{C}P^N(c)$. The main purpose of this paper is to improve this characterization. We relax the condition that $f\circ\gamma$ is a circle to the condition that it is a curve of order 2. Our main result is also an improvement of the result for $k = 1, 2$ by J.S. Pak and K. Sakamoto (see Theorem B).

2. Curves of order 2. Let M be a Riemannian manifold with Riemannian metric $\langle\ ,\ \rangle$. In order to state our result, we introduce the notion of curves of order 2. A smooth curve γ on M parametrized by its arclength s is called a *curve of order* 2 if it satisfies the following differential equation

$$\text{(2.1)}\qquad \|\nabla_{\dot\gamma}\dot\gamma\|^2\{\nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma + \|\nabla_{\dot\gamma}\dot\gamma\|^2\dot\gamma\} = \langle\nabla_{\dot\gamma}\dot\gamma, \nabla_{\dot\gamma}\nabla_{\dot\gamma}\dot\gamma\rangle\nabla_{\dot\gamma}\dot\gamma,$$

where $\nabla_{\dot\gamma}$ denotes the covariant differentiation along γ with respect to the Riemannian connection ∇ of M.

To see that the class of curves of order 2 is very wide, we recall the notion of Frenet curves. A smooth curve $\gamma = \gamma(s)$ parametrized by its arclength s is called a *Frenet curve of proper order d* if there exist orthonormal frame fields $\{V_1 = \dot\gamma, \ldots, V_d\}$ along γ and positive functions $\kappa_1(s), \ldots, \kappa_{d-1}(s)$ which satisfy the following system of ordinary differential equations

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(2.2) $$\nabla_{\dot\gamma}V_j(s) = -\kappa_{j-1}(s)V_{j-1}(s) + \kappa_j(s)V_{j+1}(s), \quad j = 1, \ldots, d,$$

where $V_0 \equiv V_{d+1} \equiv 0$. The equation (2.2) is called the Frenet formula for the Frenet curve γ. The functions $\kappa_j(s)$ $(j = 1, \ldots, d-1)$ and the orthonormal frame $\{V_1, \ldots, V_d\}$ are called the *curvatures* and the *Frenet frame* of γ, respectively.

A Frenet curve is called a *Frenet curve of order* d if it is a Frenet curve of proper order $r(\leqq d)$. For a Frenet curve of order d which is of proper order $r(\leqq d)$, we use the convention in (2.2) that $\kappa_j \equiv 0$ $(r \leqq j \leqq d-1)$ and $V_j \equiv 0$ $(r+1 \leqq j \leqq d)$. We call a curve a *helix* when all its curvatures are constant. A helix of order 1 is nothing but a geodesic. A helix of order 2, namely a curve which satisfies $\nabla_{\dot\gamma}V_1(s) = \kappa V_2(s)$, $\nabla_{\dot\gamma}V_2(s) = -\kappa V_1(s)$ and $V_1(s) = \dot\gamma(s)$, is called a *circle of curvature* κ. We regard a geodesic as a circle of null curvature. By direct calculation we know that in any Riemannian manifold every Frenet curve of order 2 satisfies the equation (2.1), so that in particular all geodesics and circles must satisfy this equation. But, in general, a curve of order 2 is not a Frenet curve of order 2. In fact, we admit the case that a curve γ of order 2 has an inflection point $\gamma(s_0)$, that is a point which satisfies $(\nabla_{\dot\gamma}\dot\gamma)(s_0) = 0$.

We now state our main result in this paper.

Theorem 1. *Let $f : M_n \to \mathbf{C}P^N(c)$ be a Kähler isometric full immersion of an n-dimensional Kähler manifold into an N-dimensional complex projective space of constant holomorphic sectional curvature c. If the image $f \circ \gamma$ of each geodesic γ on M_n is a curve of order 2 in $\mathbf{C}P^N(c)$, then one of the following holds*:

(i) *M_n is locally congruent to $\mathbf{C}P^n(c)$ and $N = n$,*
(ii) *M_n is locally congruent to $\mathbf{C}P^n(c/2)$, $N = n(n+3)/2$ and f is locally equivalent to the second Veronese embedding f_2.*

3. Veronese embeddings. It should be better to briefly recall some fundamental results on Veronese embeddings f_k $(k = 1, 2, \ldots)$ before we prove our main result (for details, see [PS]). An isometric immersion f of a Riemannian manifold M into an ambient Riemannian manifold $\widetilde{M}$ is called a *d-planar geodesic immersion* if for each geodesic γ on M the curve $f \circ \gamma$ is locally contained in a d-dimensional totally geodesic submanifold of $\widetilde{M}$. In particular, a curve ρ is called d-planar if it is locally contained in a d-dimensional totally geodesic submanifold. A d-planar curve ρ is said to be *proper d-planar* if it is not $(d-1)$-planar. We call a d-planar geodesic immersion $f : M \to \widetilde{M}$ *proper* if the curve $f \circ \gamma$ is proper d-planar for each geodesic γ on the submanifold M.

Proposition A. *The k-th Veronese embedding $f_k : \mathbf{C}P^n(c/k) \to \mathbf{C}P^N(c)$ is proper k-planar geodesic.*

In their paper [PS] J.S. Pak and K. Sakamoto considered the converse of Proposition A to obtain a characterization of each f_k:

Theorem B. *Let $f : M_n \to \mathbf{C}P^N(c)$ be a proper k-planar geodesic Kähler isometric full immersion of an n-dimensional Kähler manifold into an N-dimensional complex projective space of constant holomorphic sectional curvature c. Suppose that for each geodesic γ on M_n the curve $f \circ \gamma$ is locally contained in a k-dimensional totally real totally geodesic submanifold $\mathbf{R}P^k(c/4)$ of $\mathbf{C}P^N(c)$. Then M_n is locally congruent to $\mathbf{C}P^n(c/k)$, $N = (n+k)!/(n!k!) - 1$ and f is locally equivalent to the k-th Veronese embedding f_k.*

We remark that for each geodesic γ on $\mathbf{C}P^n(c/k)$ the curve $f_k \circ \gamma$ is a helix of proper order k in $\mathbf{R}P^k(c/4)$ with the curvatures $\kappa_1, \ldots, \kappa_{k-1}$ which are independent of the choice of γ.

The following is another (local) characterization of each Veronese embedding f_k (see [NO]). We denote by $M_n(c)$ a nonflat complex space form, which is locally complex analytically isometric to a complex projective space $\mathbf{C}P^n(c)$ when $c > 0$ or a complex hyperbolic space $\mathbf{C}H^n(c)$ when $c < 0$.

Theorem C. *Let $M_n(c)$ be a Kähler submanifold immersed in $\widetilde{M}_N(\tilde c)$. If $\tilde c > 0$ and the isometric immersion is full, then $\tilde c = kc$ and $N = (n+k)!/(n!k!) - 1$ for some positive integer k.*

4. Proof of the main result. Let γ be a Frenet curve of proper order 2 which satisfies $\nabla_{\dot\gamma}V_1(s) = \kappa(s)V_2(s)$, $\nabla_{\dot\gamma}V_2(s) = -\kappa(s)V_1(s)$ and $V_1(s) = \dot\gamma(s)$ on a Kähler manifold M with complex structure J. We set $\tau_\gamma(s) = \langle V_1(s), JV_2(s)\rangle$ and call it the *complex torsion* of γ. As we have

$$\begin{aligned}\tau'_\gamma(s) &= \nabla_{\dot\gamma}\langle V_1(s), JV_2(s)\rangle \\ &= \langle \nabla_{\dot\gamma}V_1(s), JV_2(s)\rangle + \langle V_1(s), J\nabla_{\dot\gamma}V_2(s)\rangle \\ &= \kappa(s)\cdot\langle V_2(s), JV_2(s)\rangle - \kappa(s)\cdot\langle V_1(s), JV_1(s)\rangle \\ &= 0,\end{aligned}$$

the complex torsion τ_γ is constant. The complex torsion is an important invariant for Frenet curves of proper order 2 (cf. [MO]).

An isometric immersion $f : M \to \widetilde{M}$ is said to be *isotropic* at $x \in M$ if $\|\sigma(X,X)\|/\|X\|^2 (= \lambda(x))$ is constant for each $X (\neq 0) \in T_x(M)$, where σ is the second fundamental form of f. If the immersion is isotropic at every point, then the immersion is said to be isotropic. When the function $\lambda = \lambda(x)$ is constant on M, we call M a constant (λ-)isotropic submanifold. Note that a totally umbilic immersion is isotropic, but not *vice versa*. The following is well-known ([O]).

Lemma D. *Let f be an isometric immersion of M into $(\widetilde{M}, \langle\ ,\ \rangle)$. Then f is isotropic at $x \in M$ if and only if the second fundamental form σ of f satisfies $\langle \sigma(X,X), \sigma(X,Y)\rangle = 0$ for an arbitrary orthogonal pair $X, Y \in T_x(M)$.*

We are now in a position to prove Theorem 1. It is enough to prove our theorem in case that the immersion f is not totally geodesic, namely f is not of case (i). We shall show that the submanifold M is isotropic at its each point in the ambient manifold $\mathbf{C}P^N(c)$. As at an arbitrary geodesic point of M Lemma D holds in a trivial sense, we consider a non-geodesic point $p \in M$ and a unit vector $X \in T_p(M)$ with $\sigma(X,X) \neq 0$. We take the geodesic $\gamma = \gamma(s)$ (for $s \in I$) on M with initial condition that $\gamma(0) = p$ and $\dot\gamma(0) = X$. Here, I is a sufficiently small open interval in $\mathbf{R}$ satisfying $\sigma(\dot\gamma(s), \dot\gamma(s)) \neq 0$ for all $s \in I$. By hypothesis the curve $\tilde\gamma = f \circ \gamma$ satisfies the following differential equation:

$$\text{(4.1)}\quad \|\widetilde\nabla_{\dot{\tilde\gamma}} \dot{\tilde\gamma}\|^2 \{\widetilde\nabla_{\dot{\tilde\gamma}} \widetilde\nabla_{\dot{\tilde\gamma}} \dot{\tilde\gamma} + \|\widetilde\nabla_{\dot{\tilde\gamma}} \dot{\tilde\gamma}\|^2 \dot{\tilde\gamma}\} = \langle \widetilde\nabla_{\dot{\tilde\gamma}} \dot{\tilde\gamma}, \widetilde\nabla_{\dot{\tilde\gamma}} \widetilde\nabla_{\dot{\tilde\gamma}} \dot{\tilde\gamma}\rangle \widetilde\nabla_{\dot{\tilde\gamma}} \dot{\tilde\gamma},$$

where $\widetilde\nabla_{\dot{\tilde\gamma}}$ denotes the covariant differentiation along $\tilde\gamma$ with respect to the Riemannian connection $\widetilde\nabla$ of $\mathbf{C}P^N(c)$. Put $\kappa(s) = \|\widetilde\nabla_{\dot{\tilde\gamma}(s)} \dot{\tilde\gamma}(s)\|$ for $s \in I$. In the following, we shall study on the interval I. By the Gauss formula $\widetilde\nabla_X Z = \nabla_X Z + \sigma(X,Z)$ we find that $\kappa(s) > 0$ for all $s \in I$. Note that $\langle \widetilde\nabla_{\dot{\tilde\gamma}} \dot{\tilde\gamma}, \widetilde\nabla_{\dot{\tilde\gamma}} \widetilde\nabla_{\dot{\tilde\gamma}} \dot{\tilde\gamma}\rangle = \kappa(s)\dot\kappa(s)$. Then the equation (4.1) reduces to

$$\text{(4.2)}\quad \kappa(s)\big(\widetilde\nabla_{\dot{\tilde\gamma}}(\widetilde\nabla_{\dot{\tilde\gamma}} \dot{\tilde\gamma}) + \langle \widetilde\nabla_{\dot{\tilde\gamma}} \dot{\tilde\gamma}, \widetilde\nabla_{\dot{\tilde\gamma}} \dot{\tilde\gamma}\rangle \dot{\tilde\gamma}\big) = \dot\kappa(s) \widetilde\nabla_{\dot{\tilde\gamma}} \dot{\tilde\gamma}.$$

We can set $\widetilde\nabla_{\dot{\tilde\gamma}} \dot{\tilde\gamma} = \kappa(s) V_2(s)$. Hence, from (4.2) we obtain

$$\widetilde\nabla_{\dot{\tilde\gamma}} V_2(s) = \widetilde\nabla_{\dot{\tilde\gamma}} \left(\frac{1}{\kappa(s)} \widetilde\nabla_{\dot{\tilde\gamma}} \dot{\tilde\gamma}\right) = -\kappa(s) \dot{\tilde\gamma},$$

which yields that our curve $f \circ \gamma$ is a Frenet curve of proper order 2. Since f is a Kähler immersion, we find that $\langle \dot{\tilde\gamma}, JV_2\rangle = -(1/\kappa(s))\langle J\dot\gamma, \sigma(\dot\gamma, \dot\gamma)\rangle = 0$, where J is the complex structure of $\mathbf{C}P^N(c)$. So we can take the totally real totally geodesic $\mathbf{R}P^2(c/4)$ passing p satisfying that the vectors $\dot{\tilde\gamma}(0)$ and $V_2(0)$ span the tangent space $T_p(\mathbf{R}P^2(c/4))$. We here consider the Frenet curve ρ of proper order 2 on the surface $\mathbf{R}P^2(c/4)$ passing the point $p = \rho(0)$ with the same curvature function $\kappa(s)$ and the same initial frame $\{\dot{\tilde\gamma}(0), V_2(0)\}$. Then by the uniqueness of solutions for ordinary differential equations we can see that the curve $\tilde\gamma$ locally coincides with ρ, so that $\tilde\gamma$ is locally contained in $\mathbf{R}P^2(c/4)$. The following discussion is the same as in page 40 in [PS].

As $\mathbf{R}P^2(c/4)$ is a 2-dimensional totally geodesic submanifold of $\mathbf{C}P^N(c)$, the vectors $\dot{\tilde\gamma}(s)$ and $\sigma(\dot\gamma(s), \dot\gamma(s))$ span the tangent space $T_{\tilde\gamma(s)}(\mathbf{R}P^2(c/4))$ for each s. This, together with $\widetilde\nabla_{\dot{\tilde\gamma}}(\sigma(\dot\gamma, \dot\gamma)) \in T_{\tilde\gamma(s)}(\mathbf{R}P^2(c/4))$, implies

$$\text{(4.3)}\quad \widetilde\nabla_{\dot{\tilde\gamma}}(\sigma(\dot\gamma, \dot\gamma)) = u \cdot \dot{\tilde\gamma} + v \cdot \sigma(\dot\gamma, \dot\gamma)$$

for smooth functions $u = u(s)$ and $v = v(s)$ on the interval I. Let Y be an arbitrary vector at p which is perpendicular to the vector X. We extend the vector Y to a vector field $\widetilde{Y}$ on the curve $\tilde\gamma$. The equation (4.3) gives

$$\begin{aligned}
&\langle \sigma(X,X), \sigma(X,Y)\rangle \\
&\quad = \langle \sigma(\dot\gamma, \dot\gamma), \sigma(\dot\gamma, \widetilde{Y})\rangle(0) \\
&\quad = \langle \sigma(\dot\gamma, \dot\gamma), \widetilde\nabla_{\dot{\tilde\gamma}} \widetilde{Y}\rangle(0) \\
&\quad = -\langle \widetilde\nabla_{\dot{\tilde\gamma}}(\sigma(\dot\gamma, \dot\gamma)), \widetilde{Y}\rangle(0) \\
&\quad = -\langle u(0) \cdot X + v(0) \cdot \sigma(X,X), Y\rangle = 0.
\end{aligned}$$

So it follows from Lemma D that the submanifold M is isotropic at its each point in $\mathbf{C}P^N(c)$. Moreover, due to the argument in page 41 in [PS] we know that M is a constant isotropic submanifold in the ambient space $\mathbf{C}P^N(c)$.

On the other hand we denote by R (resp. $\widetilde{R}$) the curvature tensor of M (resp. $\mathbf{C}P^N(c)$). We recall the Gauss equation

$$\begin{aligned}
&\langle \widetilde{R}(X,Y)Z, W\rangle \\
&\quad = \langle R(X,Y)Z, W\rangle + \langle \sigma(X,Z), \sigma(Y,W)\rangle \\
&\qquad - \langle \sigma(X,W), \sigma(Y,Z)\rangle.
\end{aligned}$$

Since M is a Kähler submanifold in $\mathbf{C}P^N(c)$, from this equation and

$$\widetilde{R}(X,Y)Z = \frac{c}{4}(\langle Y,Z\rangle X - \langle X,Z\rangle X + \langle JY,Z\rangle JX - \langle JX,Z\rangle JY + 2\langle X,JY\rangle JZ),$$

we find that the holomorphic sectional curvature $K(X,JX)$ of M determined by a unit vector X is given by

$$K(X,JX) = \langle R(X,JX)JX,X\rangle = c - 2\|\sigma(X,X)\|^2.$$

Thus by virtue of the above discussion M is a complex space form. Therefore from Theorem C and Proposition A we can see that M_n is locally congruent to $\mathbf{C}P^n(c/2)$, $N = n(n+3)/2$ and f is locally equivalent to the second Veronese embedding f_2.

5. Appendix. In his paper [S] Sakamoto classified 2-planar geodesic submanifolds in a complete simply connected real space form $\widetilde{M}^N(c)(= \mathbf{R}^N,\ S^N(c)$ or $H^N(c))$ of curvature c:

Theorem E. *Let $f : M^n \to \widetilde{M}^{n+p}(c)$ be a 2-planar geodesic immersion of an n-dimensional Riemannian manifold into an $(n+p)$-dimensional complete simply connected real space form $\widetilde{M}^{n+p}(c)$. Then M^n is totally umbilic in $\widetilde{M}^{n+p}(c)$ or M^n is locally congruent to a compact symmetric space of rank one embedded into some totally umbilic submanifold in $\widetilde{M}^{n+p}(c)$ through the first standard minimal embedding.*

Combining Theorem E with our discussion in this paper, we obtain the following immediately:

Theorem 2. *Let $f : M^n \to \widetilde{M}^{n+p}(c)$ be an isometric immersion of an n-dimensional Riemannian manifold into an $(n+p)$-dimensional complete simply connected real space form $\widetilde{M}^{n+p}(c)$. Suppose that for each geodesic γ on M^n the curve $f \circ \gamma$ is a curve of order 2 in the ambient space $\widetilde{M}^{n+p}(c)$. Then M^n is totally umbilic in $\widetilde{M}^{n+p}(c)$ or M^n is locally congruent to a compact symmetric space of rank one embedded into some totally umbilic submanifold in $\widetilde{M}^{n+p}(c)$ through the first standard minimal embedding.*

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