Support theorem for jump process of canonical type

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Abstract: In this paper we consider the support property and the short time aymptotic behavior of the transition density for (possibly degenerate) processes of jump type whose Lévy measure is singular.

Key words: Canonical process; finite difference operator; Lévy measure.

1. Introduction. Let an $\mathbf{R}^d$-valued jump type process $Y_t(x)$, called a *canonical* process, is given by the SDE

$$(0)\quad dY_t(x) = \sum_{j=1}^{m} X_j(Y_{t-}(x)) \circ dZ_j(t),\ Y_0(x) = x.$$

Here $z(t) = (z_1(t), \ldots, z_m(t))$ denotes an $\mathbf{R}^m$-valued Lévy process (martingale), and $X_1(x), \ldots, X_m(x)$ are smooth $\mathbf{R}^d$-valued functions on $\mathbf{R}^d$ (viewed as the vector fields), and $\circ\, dz_j(t)$ means the *canonical* (or "Stratonovich") integral. We denote by P or $P^{(x)}$ the law of $Y_{.}(x)$.

This type of jump processes was introduced by [9], and has been studied by [1], [4], [5], [6], and [8]. For this type of processes we can show, using the finite difference operator D on $L^2([0,T]\times\mathbf{R}^m, \tilde{N})$, the existence of the transition density even if the Lévy measure of the driving processes are very singular with respect to the Lebesgue measure. That is, denoting by $p_t(x,dy)$ the transition function of $Y_t(x)$, we have $p_t(x,dy) = p_t(x,y)dy$ where dy denotes the d-dimensional Lebesgue measure cf. [6] and [11]. Further we can show for a particular Lévy process $z(t)$ that *for all* $t > 0$ $\operatorname{supp} p_t(x,.) = \mathbf{R}^d$ if X_j's are not essentially degenerate cf. [5], [3].

We are interested in the description of the support of P in the Skorohod path space D. As for the characterization of the support of the law of a jump process $x_{.}(x)$, which we call *support theorems*, several results which are analogues to those in the diffusion case ([15]) have been obtained, see e.g., [5], [13].

H. Kunita [5] has given a support theorem for the canonical process (0). However, in [5], the case that the driving process consists of asymmetric stable processes (without diffusion part) having the index > 1 can not be included, since "Conditions 1,2" in [5] are not satisfied for such a case. In this note, we use a different approach, which is mainly due to [14] to cover the above case. We show, under supplementary conditions (A.1), (A.2) below, the support theorem for the canonical processes. In fact the condition (A.1) implies the "small deviations" property, which replaces Kunita's more restrictive condition.

The basic ideas of the method used here are classical estimates for the solutions of (stochastics) differential equations, and, constructions of classical trajectories which has a positive probability as a set in the path space D.

2. Processes, description of Skeltons. We will fix the time interval $[0,T]$, $T > 0$. Let $Z(t) = (Z^1(t), \ldots, Z^m(t)), 0 \le t \le T$, be a Lévy processes defined on (D, B, Π) with values in $\mathbf{R}^m$ and $Z(0) = 0$, where $D = D([0,T], \mathbf{R}^m)$ is the Skorohod space. That is, $Z^j(t) = \int_0^t \int_{\mathbf{R}} z\tilde{N}_j(dsdz)$, $j = 1, \ldots, m$, where $N_j(dsdz)$ denotes a Poisson random measure on $[0,T] \times \mathbf{R}$ and $\tilde{N}_j(dsdz) = N_j(dsdz) - E[N_j(dsdz)]$. It is decomposed as

$$\begin{aligned} Z^j(t) = &\int_0^t \int_{|z|<1} z\tilde{N}_j(dsdz) \\ &+ \Big(\int_0^t \int_{|z|\ge 1} zN_j(dsdz) \\ &- \int_0^t \int_{|z|\ge 1} zE[N_j(dsdz)] \Big) \end{aligned}$$

(a semimartingale). We assume (Z^j) are mutually independent.

We denote by $d\nu$ the Lévy measure of $Z(t)$. The measure $d\nu$ satisfies $\nu(\{0\}) = 0$ and $\int(|z|^2 \wedge$

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$1)\nu(dz) < \infty$. We assume further

$$\operatorname{supp} \nu \subset \{|z| \leq 1\},$$

but we do not assume that ν has a density with respect to the Lebesgue measure. Hence we can suppose, e.g.,

$$\nu(dz) = \sum_{n \in \mathbf{N}} k_n \delta_{a_n},$$

where $k_n > 0$, $(a_n; n \in \mathbf{N})$ is a sequence in $\mathbf{R}^m$ such that $|a_n| \to 0$ as $n \to \infty$.

We denote by $[Z, Z] \equiv ([Z^j, Z^k])$ the quadratic variation process of Z (of matrix form). We denote by $[Z]$ the scalar quadratic variation, that is, $[Z] = \sum_{j=1}^m [Z^j, Z^j]$.

Let $X_1, \ldots, X_m$ be smooth vector fields on $\mathbf{R}^d$ whose derivatives of all orders (including 0-th order) are uniformly bounded in x. We consider the canonical ("Stratonovich" type, or, Marcus equation) SDE

$$dY_t = \sum_{j=1}^m X_j(Y_t) \circ dZ^j(t), \quad Y_0 = x,$$

which defines the canonical process $Y_.$. The meaning of the equation is as follows:

$$\begin{aligned} dY_t = &\sum_{j=1}^m X_j(Y_{t-}) dZ^j(t) \\ &+ \left\{ \operatorname{Exp}\left(\sum_{j=1}^m \Delta Z^j(t) X_j \right)(Y_{t-}) \right. \\ &\left. -Y_{t-} - \sum_{j=1}^m \Delta Z^j(t) X_j(Y_{t-}) \right\} \end{aligned}$$

where $\phi(t, x) \equiv \operatorname{Exp}(tv)(x)$ is the solution flow of the differential equation

$$\frac{d\phi(t,x)}{dt} = v(\phi(t,x)), \quad \phi(0,x) = x.$$

Or equivalently (cf. [8], Lemma 2.1), we can write

$$Y_t = x + \sum_{j=1}^m X_j(Y_s) dZ^j(s) + \int_0^t h(s, ., Y_{s-}) d[Z]_s,$$

where we put

$$\begin{aligned} h(s, \omega, x) = \frac{1}{|\Delta Z(s)|^2} &\left\{ \operatorname{Exp}\left(\sum_{j=1}^m \Delta Z^j(s) X_j \right)(x) \right. \\ &\left. -x - \left(\sum_{j=1}^m \Delta Z^j(s) X_j \right)(x) \right\}, \end{aligned}$$

which is a Lipschitz process with bounded Lipschitz constant. We denote by $P^{(x)}$ the law of $Y_.$ on $D' = D([0, T], \mathbf{R}^d)$. We should be careful that $s \mapsto h(s, \omega, x)$ is not predictable.

Equation above is a coordinate free formulation of SDE with jumps for semimartingales. We shall call it a *canonical SDE driven by a vector field valued semimartingale* $Y(t) = \sum_{j=1}^m Z^j(t) X_j$ according to [4], [5], [6].

Next we construct "Skeltons". Let $u(t) = (u_1(t), \ldots, u_m(t))$, $0 \leq t \leq T$ be an $\mathbf{R}^m$-valued, piecewise smooth, cádlàg functions having finite jumps. It is decomposed as $u(t) = u^c(t) + u^d(t)$, where $u^c(t)$ is a continuous function and $u^d(t)$ is a purely discontinuous (i.e., piecewise constant except for isolated finite jumps) function.

For $\eta > 0$ we put

$$\begin{aligned} \mathcal{U}_\eta = \Big\{ &u \in D; u(t) = u^{c,\eta}(t) + u^{d,\eta}(t), \\ &\Delta u(s) \in \operatorname{supp} \nu, \\ &u^{d,\eta}(t) = \sum_{s \leq t} \Delta u(s) \cdot 1_{\{\eta < |z| \leq 1\}}(\Delta u(s)), \\ &\quad u^{c,\eta}(t) = -l_\eta \cdot t \Big\}. \end{aligned}$$

Here we put $l_\eta = \int_{\{\eta < |z| \leq 1\}} z \nu(dz)$. Put $\mathcal{U} \equiv \cup_{\eta > 0} \mathcal{U}_\eta$, and call it the space of skeltons.

Given $u = u^\eta \in \mathcal{U}_\eta$, we put a trajectory $\varphi_t^\eta \in D'$ by

$$d\varphi_t^\eta = \sum_{j=1}^m X_j(\varphi_t^\eta) \circ du_j(t), \quad \varphi_0^\eta = x.$$

The solution starting from x at $t = s$ is a (piecewise cádlàg) smooth function φ_t^η, $t \geq 0$, satisfying

$$\begin{aligned} \varphi_t^\eta = x + &\sum_{j=0}^m \int_0^t X_j(\varphi_r^\eta)(-l_\eta^{(j)}) dr \\ &+ \sum_{0 \leq r \leq t} \left\{ \operatorname{Exp}\left(\sum_{j=1}^m \Delta u_j^d(r) X_j \right)(\varphi_{r-}^\eta) - \varphi_{r-}^\eta \right\}. \end{aligned}$$

Here the integral in the second term in R.H.S. is interpreted as Stieljes integral. The function φ_t^η can be viewed as the image of a skelton u. We put

$$\mathcal{S}_\eta^x = \{\varphi_t^\eta; \varphi_t^\eta \text{ is as above, } u = u^\eta \in \mathcal{U}_\eta\}$$

and $\mathcal{S}^x = \cup_{\eta \in (0,1)} \mathcal{S}_\eta^x$.

For u, $v \in \mathcal{U}$, Skorohod metrics $\mathbf{s}$ on $D = D([0, T], \mathbf{R}^m)$ and $\mathbf{S}$ on $D' = D([0, T], \mathbf{R}^d)$ are defined as usual. The support of the Lévy process Z is defined by

$$\operatorname{supp} Z = \{u \in D\ ;\ \text{for all } \delta > 0 \quad P(\{\omega; \mathbf{s}(Z,u) < \delta\}) > 0\}.$$

The support of Y is similarly defined. By $\bar{\mathcal{S}}^x$ we denote the closure of $\mathcal{S}^x$ in $(D', \mathbf{S})$.

We define the *approximating processes* $Z^\eta(t)$, Y_t^η for each $\eta > 0$ as follows. Let $Z^\eta(t) = \int_0^t \int_{\eta<|z|\le 1} z\tilde{N}(dsdz)$, and Y_t^η is given by $dY_t^\eta = \sum_{j=1}^m X_j(Y_t^\eta) \circ dZ^{\eta,j}(t)$, $Y_0^\eta = x$. We further put the complementary process the Z^η: $\tilde{Z}^\eta(t) = \int_0^t \int_{0<|z|\le\eta} z\tilde{N}(dsdz)$ and $\tilde{Y}_t^\eta$ is given by $d\tilde{Y}_t^\eta = \sum_{j=1}^m X_j(Y_t) \circ d\tilde{Z}^{\eta,j}(t)$, $\tilde{Y}_0^\eta = x$. Note that $dZ(s) = dZ^\eta(s) + d\tilde{Z}^\eta(s)$, and $d[Z]_s = d[Z^\eta]_s + d[\tilde{Z}^\eta]_s$. Hence we have a decomposition

$$Y_t - Y_t^\eta = \sum_{j=1}^m \int_0^t \{X_j(Y_{s-}) - X_j(Y_{s-}^\eta)\} dZ_j^\eta(s) + \int_0^t \{h(s, Y_{s-}) - h^\eta(s, Y_{s-}^\eta)\} d[Z^\eta]_s + \tilde{Y}_t^\eta.$$

We now state our basic assumptions concerning "small deviations".

Setting, for every $0 < \rho < \eta$,

$$u_\rho^\eta = \int_{\rho\le|z|\le\eta} z\nu(dz),$$

we say that Z is *quasi-symmetric* if for every $\eta > 0$, there exists a sequence $\{\eta_k\}$ decreasing to 0 such that

$$|u_{\eta_k}^\eta| \longrightarrow 0 \tag{1}$$

as $k \to +\infty$. This means that for every η the compensation involved in the martingale part of $\tilde{Y}^\eta$ is somehow negligible, and of course this is true when Z is really symmetric.

To include the case where ν is not quasi-symmetric, we put the following assumption due to [14].

(A.1) (Small deviations condition) For every $\eta > 0$ such that (1) does not hold, there exists $\gamma = \gamma(\eta) > 1$ and a sequence $\{\eta_k\}$ decreasing to 0 such that as $k \to \infty$

$$\alpha_{\eta_k}^\eta = o\ (1/|u_{\eta_k}^\eta|),$$

where α_ρ^η is the smallest angle between the direction of u_ρ^η and $\operatorname{supp}\nu$ on $\{|z| = \gamma\eta\}$, when it can be defined.

Notice first that it always holds in dimension 1 (with $\alpha_{\eta_k}^\eta = 0$). Besides, it is verified in higher dimensions whenever $\operatorname{supp}\nu$ contains a sequence of spheres whose radius tend to 0 (in particular, a whole neighbourhood of 0).

For technical reasons, we also suppose that ν satisfies the following asymptotic scaling condition due to [10]:

(A.2) There exists $\beta \in [1, 2)$ and positive constants C_1, C_2 such that for any $\rho \le 1$

$$C_1\rho^{2-\beta} I \le \int_{|z|\le\rho} zz^*\nu(dz) \le C_2\rho^{2-\beta} I.$$

Besides, if $\beta = 1$, then

$$\limsup_{\eta\to 0} \left| \int_{\eta\le|z|\le 1} z\nu(dz) \right| < \infty.$$

The inequalities above stand for symmetric positive-definite matrixes. This means (A.2) demands both the non-degeneracy of the distribution of (point) masses around the origin, and, the way of concentration of masses along the radius. We notice that, if $\langle v, .\rangle$ stands for the usual scalar product with v, they are equivalent to

$$\int_{|z|\le\rho} |\langle v, z\rangle|^2 \nu(dz) \asymp \rho^{2-\beta}$$

uniformly for unit vectors $v \in S^{m-1}$. (Here $\asymp$ means the quotient of the two sides is bounded away from zero and above as $\rho \to 0$.) In particular, $\beta = \inf\{\alpha; \int_{|z|\le 1} |z|^\alpha \nu(dz) < \infty\}$ (the Blumental-Getoor index of Z), and the infimum is not reached. Notice finally that the measure ν may be very singular and have a countable support.

Theorem. *Under* (A.1), (A.2), $\operatorname{supp} Y_.(x) = \bar{\mathcal{S}}^x$. *Here* $^-$ *means the closure in* $\mathbf{S}$*-topology.*

Remark. In case $0 < \beta < 1$, the driving process $Z(t)$ is of finite variations. In this case, the correction term of $Y_.$ deriving from the exponential map vanishes. As a result, the support theorem of above form follows directly from [14], which treats the non-canonical case, without assumptions (A.1), (A.2).

3. Sketch of the proof of Theorem. The proof of the inclusion left to right is easy (cf. [14]). Hence we prove the inclusion right to left.

We prove $\bar{\mathcal{S}}^x \subset \operatorname{supp} P^{(x)}$. Since $\mathcal{S}^x = \cup_{\eta>0}\mathcal{S}_\eta^x$, it is sufficient to show for each $\eta > 0$, $\mathcal{S}_\eta^x \subset \operatorname{supp} P^{(x)}$. Let $\eta > 0$.

Let $u \in \mathcal{U}_\eta$ and $\varphi = \varphi^\eta$ be as in Section 2. We need to show for all $T > 0$, for all $\epsilon > 0$, $P(\mathbf{S}(Y, \varphi) <$

$\epsilon) > 0$.

Fix $\epsilon > 0$. Introduce $0 = t_0 < t_1 < \cdots < t_{N_T} \le T$ the jumping times of φ. Set $T_0 = 0$, and $T_1 < T_2 < \cdots$ be the jumping times of Z such that $|\Delta Z_{T_i}| > \eta$. We write $Z_i = \Delta Z_{T_i}$.

For $\rho > 0$, $\delta > 0$, we put

$$\Omega_A = \bigcap_{i=1}^{N_T} \{\{|T_i - t_i| < \rho\} \cap \{|Z_i - z_i| < \delta\}\} \bigcap \{T_{N_T+1} > T\}$$

and

$$\Omega_B = \bigcap_{i=1}^{N_T} \{\{|(T_i - T_{i-1}) - (t_i - t_{i-1})| < \rho/N_T\} \cap \{|Z_i - z_i| < \delta\}\}.$$

Then $\Omega_A \supset \Omega_B$ and Ω_B has positive $P^{(x)}$-probability, since $T_i - T_{i-1}$ follows an exponential law and z_i are in $\operatorname{supp}\nu$. Hence Ω_A has positive $P^{(x)}$-probability.

Proposition 3.1. *For each $\epsilon > 0$ there exist $\delta > 0$, $\rho > 0$ such that*

$$\mathbf{S}(Y^\eta, \varphi) < \epsilon \quad on \quad \Omega_A.$$

Proof. The proof of this proposition is long. We show we can construct concretely the trajectory Y^η which satisfies the above property. See [14] Proposition 1 for details. □

Proposition 3.2. *The right to left inclusion is shown if for all $\epsilon > 0$, there exists $c = c_\epsilon > 0$ such that*

$$P\Big(\sup_{t\le T} |\tilde{Y}_t^\eta| < \varepsilon; \Omega_B\Big) > c. \tag{2}$$

Proof. By the expression in Section 2, on $\{\sup_{t\le T} |\tilde{Y}_t^\eta| < \epsilon\} \cap \Omega_B$,

$$|Y_t - Y_t^\eta| \le \epsilon + K\int_0^t |Y_s - Y_s^\eta| d[Z^\eta]_s + K\sum_{T_i \le t} |Y_{T_i^-} - Y_{T_i^-}^\eta|.$$

By Gronwall's lemma on $[0, T_1)$ we have

$$\sup_{0\le t<T_1} |Y_t - Y_t^\eta| \le \int_0^{T_1} \epsilon e^{K(T_1 - s)} d[Z^\eta]_s < \epsilon e^{KT_1}[Z^\eta]_{T_1}.$$

Let $M > 0$ and C denotes the event $\{[Z^\eta]_{T_1} \le M\}$. By Chebyshev's inequality

$$P(C^c) = P(\{[Z^\eta]_{T_1} > M\}) \le \frac{1}{M^2}E[|[Z^\eta]_{T_1}|^2] \le \frac{1}{M^2}\int_0^{T_1}\int_{\eta\le|z|\le 1} ds|z|^2\nu(dz) \le \frac{K'}{M^2}$$

where $K' > 0$ is an absolute constant. We choose $M > 0$ so that $K'/M^2 \le c/2$.

Then $P(C) = 1 - P(C^c) \ge 1 - c/2$, hence by the assumption (2) the event $(\{\sup_{t\le T} |\tilde{Y}_t^\eta| < \epsilon\} \cap \Omega_B) \cap C$ has positive probability:

$$P\Big[\Big(\Big\{\sup_{t\le T} |\tilde{Y}_t^\eta| < \epsilon\Big\} \cap \Omega_B\Big) \cap C\Big] \ge c + \Big(1 - \frac{c}{2}\Big) - 1 = \frac{c}{2} > 0.$$

We have on this event

$$\sum_{0\le t\le T_1} |Y_t - Y_t^\eta| < \epsilon e^{KT_1} M.$$

Repeating the same argument N_T times on each $[T_i, T_{i+1})$, we get

$$\sup_{0\le t\le T} |Y_t - Y_t^\eta| < K\varepsilon.$$

Hence, by Proposition 3.1, on the event of positive probability $(\{\sup_{t\le T} |\tilde{Y}_t^\eta| < \epsilon\} \cap \Omega_B) \cap C$, we have

$$\mathbf{S}(Y, \varphi) < \epsilon$$

as desired. □

Proposition 3.3. *The condition of Proposition 3.2 holds if for any $x \in \mathbf{R}^d$, any $\epsilon > 0$ and any $t > 0$,*

$$P^{(x)}\Big(\sup_{s\le t} |\tilde{Y}_s^\eta| < \varepsilon\Big) > 0. \tag{3}$$

The proofs of this proposition and that of the assertion (3) are lengthy, and we omit the details (see [3]). The condition (3) is originally called as small deviations condition in [12]. Assumptions (A.1), (A.2) assure the condition (3). The condition (A.1), in particular, guarantees that the effect of the infinitesimal drift $\int_{|z|\le\rho} z\nu(dz)$ can be compensated by the (relatively big) jump part. In case of stable process, the condition (3) together with (A.2) implies the positivity of the density on the whole space (see [16], Theorem 1). This proves the assertion.

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