# On metaplectic representations of unitary groups: I. Splitting

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(Communicated by Shigefumi Mori, M. J. A., April 12, 2001)

**Abstract:** Model independent splittings of metaplectic representations of unitary groups are given.

**Key words:** Metaplectic representation; Weil representation; unitary group; Heisenberg group.

**0. Introduction.** Let $G$ be a unitary group of degree $n$ defined over a non-archimedean local field $F$ of characteristic different from 2. Then $G$ is embedded into the symplectic group $Sp_n(\subset GL_{2n})$. By restricting a metaplectic representation of $Sp_n$ to $G$, we obtain a projective representation $M$ of $G$. It is well-known that $M$ splits; that is, with a suitable normalizing factor $\gamma(g)$, the mapping $g \mapsto \gamma(g) \cdot M(g)$ defines a smooth representation of $G$ (cf. [Ka], [MVW]). In the study of metaplectic representations, it is often necessary to know the explicit form of $\gamma(g)$. Kudla [Ku], using results due to Rao [R] and Perrin [P], gave an explicit splitting in the case where $G$ splits over $F$ and $M$ is realized on the Schrödinger model. He also treated the non-split case by reducing it to the split case.

The object of this paper is to give an explicit splitting of $M$ available in both split and non-split cases in a uniform way. Our splitting relies on a realization of $M$ given in [MVW], which is naturally constructed from an irreducible smooth representation $\rho$ of the Heisenberg group and essentially independent of the choice of a model of $\rho$. Thus our splitting is, in a sense, *model-independent*. We note that the result has been proved in [MS] in the case $n = 1$.

The paper is organized as follows. In §1, after giving some notations and recalling a realization of metaplectic representations after [MVW], we state the main result of the paper (Theorem 1.8). In §2, we prove the theorem by calculating the cocycles of $M$ explicitly.

**1. Main result. 1.1.** Let $F$ be a non-archimedean local field of characteristic different from 2 and $K$ a semisimple commutative algebra over $F$ with $\dim_F K = 2$. Then $K$ is either a quadratic extension of $F$ or isomorphic to $F \oplus F$. In the latter case, we fix an isomorphism $K \simeq F \oplus F$ to identify $K$ with $F \oplus F$. Denote by $\omega$ the quadratic character of $F^\times$ corresponding to $K/F$ by local class field theory. Let $\mathcal{O}_F$ be the integer ring of $F$ and

$$\mathcal{O}_K = \begin{cases} \text{the integer ring of } K \cdots K \text{ is a field} \\ \mathcal{O}_F \oplus \mathcal{O}_F \qquad\qquad \cdots K = F \oplus F. \end{cases}$$

For $z \in K$, we put $\mathrm{Tr}_{K/F}(z) = z + z^\sigma$, $N_{K/F}(z) = zz^\sigma$ and $|z|_K = |N_{K/F}(z)|_F$, where $\sigma$ denotes the nontrivial automorphism of $K/F$ and $|\cdot|_F$ the normalized valuation of $F$. For $A \in M_{mn}(K)$, we put $A^* = {}^tA^\sigma$. By a *lattice* of a finite dimensional vector space $W$ over $K$, we always mean an $\mathcal{O}_F$-lattice of $W$.

**1.2.** Let $W = K^n$ be the vector space of $n$-column vectors in $K$. We fix a $Q \in GL_n(K)$ with $Q^* = -Q$ and define a nondegenerate $F$-valued alternating form $\langle\ ,\ \rangle$ on $W$ by $\langle w, w'\rangle = \mathrm{Tr}_{K/F}(w^*Qw')$ $(w, w' \in W)$. Let $H$ be the Heisenberg group associated with the symplectic space $(W, \langle\ ,\ \rangle)$. By definition, the underlying set of $H$ is $W \times F$ and the multiplication is given by $(w, x)(w', x') = (w+w', x+x' + \langle w, w'\rangle/2)$. Let $G = U(Q) = \{g \in GL_n(K) \mid g^*Qg = Q\}$ be the unitary group of $Q$. Then $G$ acts on $H$ by $g \cdot (w, x) = (gw, x)$ $(g \in G, (w, x) \in H)$.

**1.3.** From now on, we fix a nontrivial additive character $\psi$ of $F$. Let $(\rho, V)$ be a smooth irreducible representation of $H$ such that $\rho((0, x)) = \psi(x) \cdot \mathrm{Id}_V$ $(x \in F)$. By the Stone–von Neumann theorem, for each $g \in G$, there exists an automorphism $M(g)$ of $V$ satisfying

$$\text{(1.1)}\quad M(g)\rho(h)M(g)^{-1} = \rho(g \cdot h) \qquad (h \in H)$$

and $g \mapsto M(g)$ defines a projective representation

1991 Mathematics Subject Classification. 11F27, 22E50.

of $G$ on $V$ (a *metaplectic representation* of $G$). To simplify the notation, we write $\rho(w,x)$ for $\rho((w,x))$.

**1.4.** We next recall a realization of $M(g)$ attached to $(\rho, V)$ given in [MVW]. Let $g \in G$ ($g \neq 1$) and put $W_g = W/\operatorname{Ker}(g-1)$. Let $d_g w$ be the Haar measure on $W_g$ self-dual with respect to the pairing $(w,w') \mapsto \psi(\langle w, (g-1)w'\rangle)$. For each $v \in V$, there exist a lattice $L_v$ of $W_g$ and $v' \in V$ satisfying the following condition; for any lattice $L$ of $W_g$ containing $L_v$, we have

$$v' = \int_L \psi\Big(\frac{1}{2}\langle w, gw\rangle\Big)\rho((1-g)w, 0)v\, d_g w.$$

We put $M(g)v = v'$. If $g = 1$, we set $M(g) = \mathrm{Id}_V$. Then $M(g)\colon G \to \operatorname{End}(V)$ satisfies (1.1) and $M(g) \circ M(g^{-1}) = \mathrm{Id}_V$ holds for any $g \in G$ (see [MVW, Ch. 2, II.2–4]).

**1.5.** To recall a definition of Weil constants (cf. [W]), let $d_K w$ be the Haar measure on $K$ self-dual with respect to the pairing $(w,w') \mapsto \psi(\mathrm{Tr}_{K/F}(w^\sigma w'))$, and $\mathcal{S}(K)$ the space of locally constant and compactly supported functions on $K$. Denote by $\widehat{f}$ the Fourier transform of $f \in \mathcal{S}(K)$:

$$\widehat{f}(w) = \int_K f(w')\psi(\mathrm{Tr}_{K/F}(w^\sigma w'))d_K w'.$$

Then there exists a nonzero complex number $\lambda_K(\psi)$ such that the following equality holds for any $f \in \mathcal{S}(K)$ and $a \in F^\times$:

$$\text{(1.2)}\quad \int_K f(w)\psi(aww^\sigma)d_K w = \lambda_K(\psi)\omega(a)|a|_F^{-1}\int_K \widehat{f}(w)\psi(-a^{-1}ww^\sigma)d_K w.$$

It is known that $\lambda_K(\psi)^2 = \omega(-1)$.

**1.6.** Let $R \in M_n(K) - \{0\}$ and put $\operatorname{Ker}(R) = \{w \in W \mid Rw = 0\}$ and $n(R) = \dim_K W/\operatorname{Ker}(R)$. Suppose that $\operatorname{Ker}(R) = \operatorname{Ker}(R^*)$. Then there exists an $A \in GL_n(K)$ such that $A^*RA = \begin{pmatrix} R_0 & 0 \\ 0 & 0\end{pmatrix}$ with $R_0 \in GL_{n(R)}(K)$. We set $\Delta(R) = \det(R_0) \in K^\times/N_{K/F}(K^\times)$, which is independent of the choice of $A$. Note that $\Delta(R) = \det\big((w_i^* R w_j)_{1\le i,j\le n(R)}\big)$, where $\{w_1, \dots, w_{n(R)}\}$ is a $K$-basis of $W/\operatorname{Ker}(R)$. We put $n(R) = 0$ and $\Delta(R) = 1$ if $R$ is the zero matrix.

**1.7.** Let $g \in G$ and put $R_g = Q(g-1)$. Then we have $\operatorname{Ker}(R_g) = \operatorname{Ker}(R_g^*) = \operatorname{Ker}(g-1)$. Set $\nu_g = n(R_g) = \dim_K W/\operatorname{Ker}(g-1)$ and $\xi_g = \Delta(R_g)$. Let $\mathcal{X}$ be the set of unitary characters $\chi$ of $K^\times$ with $\chi|_{F^\times} = \omega$. For $\chi \in \mathcal{X}$, we put

$$\text{(1.3)}\qquad \gamma_\chi(g) = \lambda_K(\psi)^{-\nu_g}\chi(\xi_g).$$

Since $\chi$ is trivial on $N_{K/F}(K^\times)$, $\gamma_\chi(g)$ is well-defined. It is easily verified that $\gamma_\chi(g)\gamma_\chi(g^{-1}) = 1$. Set $\mathcal{M}_\chi(g) = \gamma_\chi(g)M(g)$. We are now able to state the main result of the paper:

**1.8. Theorem.** *For $\chi \in \mathcal{X}$, the mapping $g \mapsto \mathcal{M}_\chi(g)$ defines a smooth representation of $G$ on $V$.*

**Remark 1.** Theorem 1.8, with a suitable modification, also holds for symplectic groups and quaternion unitary groups. (In the symplectic case, we obtain a projective representation whose cocyles are valued in $\{\pm 1\}$.) Theorem 1.8 also holds in the archimedean case.

**Remark 2.** A straightforward calculation shows that our splitting coincides with the one given in [Ku] in the case where $G$ splits over $F$ and $(\rho, V)$ is the Schrödinger model.

**2. Proof of the main result. 2.1.** Let $S$ be a Hermitian matrix of degree $n$. Note that $\Delta(S) \in F^\times/N_{K/F}(K^\times)$. Let $d_S w$ be the Haar measure on $W_S = W/\operatorname{Ker}(S)$ self-dual with respect to the pairing $(w,w') \mapsto \psi(\mathrm{Tr}_{K/F}(w^* S w'))$. The following well-known fact is an immediate consequence of (1.2).

**2.2. Lemma.** *There exists a lattice $L_S$ of $W_S$ such that, for any lattice $L$ of $W_S$ containing $L_S$, we have*

$$\int_L \psi(w^* S w)d_S w = \lambda_K(\psi)^{n(S)}\omega(\Delta(S)).$$

**2.3.** For $g, g' \in G$, let $c(g,g') \in \mathbf{C}^\times$ be the cocycle of $M$ given by

$$M(g)M(g') = c(g,g')\cdot M(gg').$$

By an argument similar to [P, §1.4], we see that the proof of Theorem 1.8 is reduced to that of the following fact.

**2.4. Lemma.** *Let $g, g' \in G$ and suppose that*

$$\text{(2.1)}\quad g \neq 1, \det(g'-1) \neq 0, \det(gg'-1) \neq 0.$$

*Then, for $\chi \in \mathcal{X}$, we have*

$$c(g,g') = \frac{\gamma_\chi(gg')}{\gamma_\chi(g)\gamma_\chi(g')}.$$

**2.5.** To show Lemma 2.4, we henceforth fix $g, g' \in G$ satisfying (2.1) and put $S = Q(g'-1)(gg'-1)^{-1}(g-1)$. Since $S = 2^{-1}(QB - B^*Q)$ with $B = g + (g^{-1}-1)(gg'-1)^{-1}(g-1)$, we have $S^* = S$. Note that $\operatorname{Ker}(S) = \operatorname{Ker}(g-1)$.

**2.6. Lemma.** *We have*

$$c(g,g') = \lambda_K(\psi)^{\nu_g} \cdot \omega(\Delta(S)).$$

*Proof.* Let $v \in V - \{0\}$. Taking sufficiently large lattices $L$ and $L'$ of $W_g$ and $W$ respectively, we have

$$\begin{aligned}&M(g)M(g')v\\&= \int_L d_g w \int_{L'} d_{g'} w' \psi\Big(\frac{1}{2}\langle w, gw\rangle + \frac{1}{2}\langle w', g'w'\rangle\\&+\frac{1}{2}\langle (1-g)w, (1-g')w'\rangle\Big)\\&\rho((1-g)w + (1-g')w', 0)v.\end{aligned}$$

We may (and do) assume that $\pi(g'L') \subset L$, where $\pi\colon W \to W_g$ denotes the natural projection. Changing the variable $w$ into $w + g'w'$, we obtain

$$\begin{aligned}&M(g)M(g')v\\&= \int_{L'} \psi\Big(-\frac{1}{2}\langle w', (1-gg')w'\rangle\Big) f(w') d_{g'}w',\end{aligned}$$

where

$$\begin{aligned}f(w') = \int_L \psi\Big(&\frac{1}{2}\langle w, (g-1)w\rangle\\&+\frac{1}{2}\langle w, (1-g^{-1})(1+gg')w'\rangle\Big)\\&\rho((1-g)w + (1-gg')w', 0)v d_g w.\end{aligned}$$

A standard argument shows that the mapping $f\colon W \to V$ is compactly supported. We thus have

$$\begin{aligned}&M(g)M(g')v = \int_W d_{g'}w' \int_L d_g w\\&\psi\Big(-\frac{1}{2}\langle w', (1-gg')w'\rangle + \frac{1}{2}\langle w, (g-1)w\rangle\\&+\frac{1}{2}\langle (1-g)w, (1+gg')w'\rangle\Big)\\&\rho((1-g)w + (1-gg')w', 0)v.\end{aligned}$$

Changing the variable $w'$ into $w' - (1-gg')^{-1}(1-g)w$, we obtain

$$\begin{aligned}&M(g)M(g')v\\&= \int_L \psi\Big(\frac{1}{2}\langle w, (g'-1)(gg'-1)^{-1}(g-1)w\rangle\Big) d_g w\\&\times \int_{L'} \psi\Big(\frac{1}{2}\langle w', gg'w'\rangle\Big) \rho((1-gg')w', 0)v d_{g'}w'\\&= \int_L \psi(w^*Sw) d_g w \cdot \frac{d_{g'}w'}{d_{gg'}w'} M(gg')v.\end{aligned}$$

This implies

$$\begin{aligned}c(g,g') &= \frac{d_{g'}w'}{d_{gg'}w'} \cdot \frac{d_g w}{d_S w} \int_L \psi(w^*Sw) d_S w\\&= \lambda_K(\psi)^{\nu_g} \cdot \omega(\Delta(S))\end{aligned}$$

as claimed. □

**2.7.** For $a, b \in K^\times$, we write $a \sim b$ if $ab^{-1} \in N_{K/F}(K^\times)$. To prove Lemma 2.4, it now remains to show the following:

**2.8. Lemma.**

$$\Delta(S) \sim \frac{\xi_g \xi_{g'}}{\xi_{gg'}}.$$

*Proof.* Set $Y = Q(g'-1)(gg'-1)^{-1}Q^{-1}$ and $X = Q(g-1)$, and take an element $A$ of $GL_n(K)$ such that $X' = A^*XA = \mathrm{diag}(x, 0_{n-\nu_g})$ with $x \in GL_{\nu_g}(K)$. Note that $S = YX$ and $\xi_g = \det x$. Let

$$Y' = A^*Y(A^*)^{-1} = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix}, S' = A^*SA = Y'X'$$

with $y_1$ of size $\nu_g$ and $y_4$ of size $n - \nu_g$. Since

$$S' = \begin{pmatrix} y_1 x & 0 \\ y_3 x & 0 \end{pmatrix}$$

and $S'^* = S'$, we have $y_3 = 0$. We next show that $y_4$ is the identity matrix. Let $u$ be any element of $K^{n-\nu_g}$ (a row vector) and $\mathbf{0} = (0, \ldots, 0) \in K^{\nu_g}$. Then we have $[\mathbf{0}u]Y' = [\mathbf{0}\, uy_4]$. Observe that $[\mathbf{0}u]A^*Q = [\mathbf{0}u]A^*Qg$, since $[\mathbf{0}u]X'$ is the zero vector. It follows that

$$\begin{aligned}[\mathbf{0}u]Y' &= [\mathbf{0}u]A^*Q(g'-1)(gg'-1)^{-1}Q^{-1}(A^*)^{-1}\\&= [\mathbf{0}u]A^*Q(gg'-g)(gg'-1)^{-1}Q^{-1}(A^*)^{-1}\\&= [\mathbf{0}u] - [\mathbf{0}u]A^*Q(g-1)(gg'-1)^{-1}Q^{-1}(A^*)^{-1}\\&= [\mathbf{0}u],\end{aligned}$$

which implies $y_4 = 1_{n-\nu_g}$ as claimed. We thus have $\det Y = \det Y' = \det y_1$ and $\Delta(S) \sim \Delta(S') = \det y_1 \cdot \det x = \det Y \cdot \xi_g \sim \det(g'-1)\det(gg'-1)^{-1} \cdot \xi_g$. The proof of the lemma is now complete since $\xi_{g'} = \det Q \cdot \det(g'-1)$ and $\xi_{gg'} = \det Q \cdot \det(gg'-1)$. □

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