Greenberg's conjecture for Dirichlet characters of order divisible by p

By Takae Tsuji

Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1, Komaba, Meguro-ku, Tokyo 153-8914

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Abstract: Fix an odd prime number p. For an even Dirichlet character χ, it is conjectured that the Iwasawa λ-invariant $\lambda_{p,\chi}$ related to the χ-part of ideal class group is zero ([5], [2]). In this note, we show (under some assumptions) that there exist infinitely many characters χ of order divisible by p for which the conjecture is true by using Kida's formula ([6]).

Key words: Iwasawa theory; Greenberg's conjecture; Kida's formula.

For a prime number p and a number field k, let k_∞/k be the cyclotomic $\mathbf{Z}_p$-extension with its n-th layer k_n. We denote by A_n the p-Sylow subgroup of the ideal class group of k_n for each $n \geq 0$ and put $X_\infty = \varprojlim A_n$ where the projective limit is taken with respect to the relative norms. It is conjectured that X_∞ is a finite abelian group if k is totally real([2], [5, p. 316]), which is often called Greenberg's conjecture.

When k is a real abelian field, decomposing X_∞ by the action of $\Delta = \mathrm{Gal}(k_\infty/\mathbf{Q}_\infty)$, we can formulate Greenberg's conjecture for (p, χ) for each character χ of Δ (see below). In [7], when the order of χ is divisible by p and χ satisfies some assumptions, the author gave a sufficient (but not necessary) condition for the conjecture for (p, χ) to be true (Proposition 1). In this note, we will rewrite this sufficient condition by using Kida's formula for p-adic L-function proved by Sinnott [6]. Furthermore, we will show that there exist infinitely many characters χ which satisfy the condition (Proposition 5).

In the following, we fix an odd prime number p. Let χ be a $\overline{\mathbf{Q}}_p^\times$-valued nontrivial even primitive Dirichlet character of the first kind, i.e. the conductor of χ is not divisible by p^2. Let k be the real abelian field corresponding to χ. Since χ is of the first kind, we have $\mathrm{Gal}(k/\mathbf{Q}) \cong \Delta$, and hence X_∞ becomes a $\mathbf{Z}_p[\Delta]$-module. We define V^χ by

$$V^\chi := \{x \in X_\infty \otimes_{\mathbf{Z}_p} \Phi \mid \delta x = \chi(\delta)x, \forall \delta \in \Delta\},$$

where Φ denotes the field generated by the values of χ over $\mathbf{Q}_p$. It is known that V^χ is a finite dimensional Φ-vector space (cf. [5, Theorem 5]). We put $\lambda_\chi = \lambda_{p,\chi} = \dim_\Phi V^\chi$. We know that X_∞ is finitely generated over $\mathbf{Z}_p$ ([1]). Then Greenberg's conjecture for (p, χ) is stated as follows:

$$\lambda_\chi = 0.$$

Let $L_p(s, \chi)$ denote the Kubota-Leopoldt p-adic L-function associated to χ. By Iwasawa, it is shown that there exists a unique power series $g_\chi(T)$ with coefficients in $\mathcal{O}$, the integer ring of Φ, such that

$$g_\chi((1+p)^s - 1) = L_p(1-s, \chi)$$

(cf. [9, Theorem 7.10]). By using Ferrero-Washington theorem([1]), we can define $\lambda_\chi^* = \min\{n \mid a_n \in \mathcal{O}^\times\}$, where $g_\chi(T) = \sum a_n T^n$. It follows from the Iwasawa main conjecture proved in [3] that

$$\lambda_\chi \leq \lambda_\chi^*$$

(cf. e.g. [7, § 3]). Hence, if $\lambda_\chi^* = 0$, we clearly have $\lambda_\chi = 0$.

When $\chi\omega^{-1}(p)$ is in $\boldsymbol{\mu}_{p^\infty}$, the group of all p-power roots of unity, we have $\lambda_\chi^* \geq 1$ by the formula for $L_p(0, \chi)$ (cf. [9, Theorem 5.11]) where ω denotes the Teichmüller character. However, Ichimura and Sumida showed that $\lambda_\chi \leq \lambda_\chi^* - 1$ in the case where $\chi\omega^{-1}(p) = 1$ ([4, $(5'_B)$ p.724, Remark 5]). Hence, in this case, we have $\lambda_\chi = 0$ if $\lambda_\chi^* = 1$. For other cases, the author proved the following:

Proposition 1 [7, Proposition 2.3]. *Assume that $\chi\omega^{-1}(p) \in \boldsymbol{\mu}_{p^\infty}$ and $\chi\omega^{-1}(p) \neq 1$. We further assume that $\lambda_\chi^* = 1$ or that $B_{1,\chi\omega^{-1}}$ is a p-unit. Then we have $\lambda_\chi = 0$. Here $B_{1,\chi\omega^{-1}}$ denotes the generalized first Bernoulli number.*

In this note, we concentrate on the case where $\chi\omega^{-1}(p) \in \boldsymbol{\mu}_{p^\infty}$ and $\chi\omega^{-1}(p) \neq 1$, and consider the condition that $\lambda_\chi^* = 1$ (resp. $B_{1,\chi\omega^{-1}}$ is a p-unit).

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We write

$$\chi = \psi\rho$$

where the order of ψ (resp. ρ) is prime to p (resp. a p-power). We note that if $\chi\omega^{-1}(p) \in \boldsymbol{\mu}_{p^\infty}$ and $\chi\omega^{-1}(p) \neq 1$, we have $\psi\omega^{-1}(p) = 1$ and $\rho(p) \neq 1$ (in particular ρ is non-trivial). Then the following is known:

Lemma 2. *Let χ be an even Dirichlet character of the first kind. We write $\chi = \psi\rho$ as above. Then the following hold*:

(i) (*Sinnott* [6, Theorem 2.1]) *Let N be the number of places v on $\mathbf{Q}_\infty$ satisfying $\rho(l) = 0$ and $\psi\omega^{-1}(l) = 1$ where l is the prime number below v. Then we have*

$$\lambda^*_\chi = \lambda^*_\psi + N.$$

(ii) *We have the following congruence*

$$B_{1,\chi\omega^{-1}} \equiv \Bigl(\prod_l (1 - \psi\omega^{-1}(l))\Bigr) B_{1,\psi\omega^{-1}} \mod \pi$$

where l runs over all prime numbers such that $\rho(l) = 0$ and π denotes a prime element of $\mathcal{O}$.

Proof of (ii). Although we can show the assertion (ii) in the same way as in the proof of (i) in [6], we give its proof for the convenience of the reader. For the properties of the generalized Bernoulli number, see [9, § 4.1]. Let m be the conductor of $\chi\omega^{-1}$. We have

$$\begin{aligned} B_{1,\chi\omega^{-1}} &= \frac{1}{m}\sum_{a=1}^{m} \chi\omega^{-1}(a)a \\ &= \frac{1}{m}\sum_{\substack{a=1\\(a,m)=1}}^{m} \psi\omega^{-1}(a)\rho(a)a \\ &\equiv \frac{1}{m}\sum_{\substack{a=1\\(a,m)=1}}^{m} \psi\omega^{-1}(a)a \mod \pi. \end{aligned}$$

On the other hand, we have

$$\frac{1}{m}\sum_{\substack{a=1\\(a,m)=1}}^{m} \psi\omega^{-1}(a)a = \Bigl(\prod_{l|m} (1 - \psi\omega^{-1}(l))\Bigr) B_{1,\psi\omega^{-1}},$$

where l runs over all prime divisors of m. This proves the assertion (ii). □

By the above lemma, we can rewrite the sufficient condition for $\lambda_\chi = 0$ in Proposition 1 as follows:

Lemma 3. *Assume $\chi\omega^{-1}(p) \in \boldsymbol{\mu}_{p^\infty}$ and $\chi\omega^{-1}(p) \neq 1$. We write $\chi = \psi\rho$ as above. The following hold*:

(i) *$\lambda^*_\chi = 1$ if and only if $\lambda^*_\psi = 1$ and $\psi\omega^{-1}(l) \neq 1$ for any prime number l such that $\rho(l) = 0$.*

(ii) *$B_{1,\chi\omega^{-1}}$ is a p-unit if and only if $B_{1,\psi\omega^{-1}}$ is a p-unit and $\psi\omega^{-1}(l) \neq 1$ for any prime number l such that $\rho(l) = 0$.*

Using Chebotarev density theorem, we will show the following:

Lemma 4. *Let ψ be an even Dirichlet character of the first kind of order prime to p which is distinct from a power of ω. Let r and s be integers such that $r \geq s \geq 0$ and $r \geq 1$. Then there exist infinitely many characters ρ such that*

(a′) *ρ is of the first kind of order p^r,*

(b′) *$\rho(p)$ is a primitive p^s-th root of unity,*

(c′) *$\psi\omega^{-1}(l) \neq 1$ for any prime number l such that $\rho(l) = 0$.*

By Proposition 1, Lemmas 3 and 4, we obtain the following.

Proposition 5. *Let ψ be an even Dirichlet character of the first kind of order prime to p such that $\psi\omega^{-1}(p) = 1$ and $r \geq s \geq 1$ integers. We assume that $\lambda^*_\psi = 1$ or $B_{1,\psi\omega^{-1}}$ is a p-unit. Then there exist infinitely many even characters χ such that*

(a) *$\chi = \psi\rho$ with a character ρ of the first kind of order p^r,*

(b) *$\chi\omega^{-1}(p)$ is a primitive p^s-th root of unity,*

(c) *$\lambda_\chi = 0$.*

Indeed, let ρ be a character satisfying the conditions (a′), (b′) and (c′) in Lemma 4 and put $\chi = \psi\rho$. Then the condition (b′) implies (b) in Proposition 5. Combining the condition (c′), the assumption that $\lambda^*_\psi = 1$ (resp. $B_{1,\psi\omega^{-1}}$ is a p-unit) and Lemma 3, we have $\lambda^*_\chi = 1$ (resp. $B_{1,\chi\omega^{-1}}$ is a p-unit). Thus, by Proposition 1, we obtain $\lambda_\chi = 0$.

Proof of Lemma 4. We give a proof when $s \geq 1$. Let F be the abelian field corresponding to $\psi\omega^{-1}$, and we put $L = \mathbf{Q}(\zeta_{p^r}, p^{1/p^{r-s+1}})$ and $L' = \mathbf{Q}(\zeta_{p^r}, p^{1/p^{r-s}})$. Here ζ_m denotes a primitive m-th root of unity for any integer $m \geq 1$. By the assumption that ψ is distinct from a power of ω, we have $F \not\subset L'$. Thus we can take $\delta \in \mathrm{Gal}(FL/\mathbf{Q})$ satisfying $\delta|_F \neq 1$, $\delta|_L \neq 1$ and $\delta|_{L'} = 1$. Let $\mathfrak{l}$ be a prime of FL such that the Frobenius $\delta_\mathfrak{l}$ of $\mathfrak{l}$ in $\mathrm{Gal}(FL/\mathbf{Q})$ coincides with δ and l the prime number below $\mathfrak{l}$. Chebotarev density theorem guarantees the existence of infinitely many such l. By $\delta_\mathfrak{l}|_L \neq 1$ and $\delta_\mathfrak{l}|_{L'} = 1$, we obtain $l \equiv 1 \bmod p^r$, $p \in ((\mathbf{Z}/l\mathbf{Z})^\times)^{p^{r-s}}$ and $p \notin ((\mathbf{Z}/l\mathbf{Z})^\times)^{p^{r-s+1}}$, that is, $l \equiv 1 \bmod p^r$ and $p^{r-s} \parallel [(\mathbf{Z}/l\mathbf{Z})^\times : \langle p\rangle]$. Let $k^{(l)}$

be the cyclic extension of $\mathbf{Q}$ of degree p^r contained in $\mathbf{Q}(\zeta_l)$ and ρ_l a Dirichlet character corresponding to $k^{(l)}$. Then ρ_l satisfies (a′). By using a canonical isomorphism from $(\mathbf{Z}/l\mathbf{Z})^\times$ to $\mathrm{Gal}(\mathbf{Q}(\zeta_l)/\mathbf{Q})$, we have $p^{r-s} \parallel [(\mathbf{Z}/l\mathbf{Z})^\times : \langle p \rangle]$ if and only if the order of the decomposition group of p in $\mathrm{Gal}(k^{(l)}/\mathbf{Q})$ is p^s, i.e., $\rho_l(p)$ is a primitive p^s-th root of unity. Hence ρ_l satisfies (b′). On the other hand, by $\delta_{\mathfrak{l}}|_F \neq 1$, we have $\psi\omega^{-1}(l) \neq 1$, that is, ρ_l satisfies (c′).

For the case where $s = 0$, one can show the assertion as above. □

In conclusion, we remark on the case $\chi\omega^{-1}(p) \notin \boldsymbol{\mu}_{p^\infty}$ (resp. $\chi\omega^{-1}(p) = 1$). It is known and follows immediately from the formula for $L_p(0,\chi)$ (cf. [9, Theorem 5.11]) that $\lambda_\chi^* = 0$ if and only if $B_{1,\chi\omega^{-1}}$ is a p-unit when $\chi\omega^{-1}(p) \notin \boldsymbol{\mu}_{p^\infty}$. Further, we can show that $\lambda_\chi = 0$ if $B_{1,\chi\omega^{-1}}$ is a p-unit even when $\chi\omega^{-1}(p) = 1$ (cf. e.g. [8]). Thus, by Lemmas 2, 4 and the comment above Proposition 1, we obtain the following:

Proposition 6. *Let ψ be an even Dirichlet character of the first kind of order prime to p which is distinct from a power of ω and $r \geq 1$ an integer. We assume that $B_{1,\psi\omega^{-1}}$ is a p-unit (resp. $\lambda_\psi^* = 1$ or $B_{1,\psi\omega^{-1}}$ is a p-unit) if $\psi\omega^{-1}(p) \neq 1$ (resp. $\psi\omega^{-1}(p) = 1$). Then there exist infinitely many even characters χ such that*

(a) *$\chi = \psi\rho$ with a character ρ of the first kind of order p^r,*

(b) *$\chi\omega^{-1}(p) = 1$ if $\psi\omega^{-1}(p) = 1$,*

(c) *$\lambda_\chi = 0$.*

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