On some Pachpatte integral inequalities involving convex functions

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Abstract: In the present paper we establish some new integral inequalities involving convex function as a certain extensions of Pachpatte's inequality by using a fairly elementary analysis.

Key words: Integral inequality; convex function; Pachpatte's integral inequality.

1. Introduction. Let $f, g : [a, b] \to R$ be convex mapping. For two elements x, y in $[a, b]$, we shall define the mappings $F(x, y)$, $G(x, y) : [0, 1] \to R$ as follows:

$$\text{(1.1)} \quad F(x,y)(t) = \frac{1}{2}\Big(f\big(tx+(1-t)y\big)+f\big((1-t)x+ty\big)\Big),$$

$$\text{(1.2)} \quad G(x,y)(t) = \frac{1}{2}\Big(g\big(tx+(1-t)y\big)+g\big((1-t)x+ty\big)\Big).$$

In [2] Dragomir and Ionescu established some interesting properties of such mappings. In particular, in [2], it is shown that $F(x, y)$, $G(x, y)$ are convex on $[a, b]$. In another paper [7], Pečarić and Dragomir proved that following statements are equivalent for mapping $f, g : [a, b] \to R$:

(i) f, g are convex on $[a, b]$;

(ii) for all $x, y \in [a, b]$ the mappings $f_0, g_0 : [0, 1] \to R$ defined by $f_0(t) = f\big(tx + (1-t)y\big)$ or $f\big((1-t)x+ty\big)$, $g_0(t) = g\big(tx+(1-t)y\big)$ or $g\big((1-t)x+ty\big)$ are convex on $[0, b]$.

Form these properties, it is easy to observe that f_0 and g_0 are convex on $[0, 1]$, for all x, $y \in [a, b]$ the mappings $F_0(x, y)$, $G_0(x, y) : [0, 1] \to R$ defined by

$$\text{(1.3)} \quad F_0(x,y)(\lambda) = \frac{1}{2}\Big[f\Big(\lambda\big[tx+(1-t)y\big]+(1-\lambda)\big[(1-t)x+ty\big]\Big) + f\Big((1-\lambda)\big[tx+(1-t)y\big]+\lambda\big[(1-t)x+ty\big]\Big)\Big],$$

$$\text{(1.4)} \quad G_0(x,y)(\lambda) = \frac{1}{2}\Big[g\Big(\lambda\big[tx+(1-t)y\big]+(1-\lambda)\big[(1-t)x+ty\big]\Big) + g\Big((1-\lambda)\big[tx+(1-t)y\big]+\lambda\big[(1-t)x+ty\big]\Big)\Big],$$

are convex on $[0, 1]$, f_0 and g_0 are integrable on $[0, 1]$ and hence f_0g_0 is also integrable on $[0, 1]$. Similarly, If f and g are convex on $[a, b]$, they are integrable on $[a, b]$ and hence fg is also integrable on $[a, b]$. Consequently, it is easy to see that if f and g are convex on $[a, b]$, then $F_0 = F_0(x, y)$ and $G_0 = G_0(x, y)$ and hence F_0g, G_0g, F_0f, G_0f are also integrable on $[a, b]$. We shall use these facts in our discussion without further mention.

Recently, Pachpatte [6] established some new integral inequalities involving the functions $F(x, y)$ and $G(x, y)$ as defined in (1.1) and (1.2). This paper deals with some new generalizations of the Pachpatte's inequalities, using the functions F_0 and G_0 as defined in (1.3) and (1.4). The analysis used in the proof is elementary and we believe that the inequalities established here are of independent interest. For other results related to such inequalities, please see [1–7] where further references are given.

2. Main results. We need the inequalities in the following lemma, which are appear in the proof of Theorem 1 of the paper [6].

Lemma 2.1. *The assumptions that f and g are nonnegative and convex, imply that we may assume that f, $g \in C^1$ and that we have the following estimates*

$$f(tx + (1-t)y) \geq f(x) + (1-t)(y-x)f'(x),$$
$$f((1-t)x + ty) \geq f(x) + t(y-x)f'(x),$$
$$g(tx + (1-t)y) \geq g(x) + (1-t)(y-x)g'(x),$$
$$g((1-t)x + ty) \geq g(x) + t(y-x)f'(x),$$

for x, $y \in [a, b]$ and $t \in [0, 1]$.

The main results on integral inequalities are presented as follows:

Theorem 2.2. *Let f and g be real-valued, nonnegative and convex function on $[a, b]$ and map-*

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pings $F_0(x,y)$ and $G_0(x,y)$ be defined by (1.3) *and* (1.4). *Then for all* t, λ *in* $[0,1]$ *we have*

$$(2.1)\quad (3-t)\int_a^b f(y)g(y)(b-y)\,dy \le \int_a^b\Big(\int_a^b\Big(F_0(x,y)(\lambda)g(x)+G_0(x,y)(\lambda)f(x)\Big)dx\Big)dy + \frac{1}{2}(1-t)g(a)f(a)(b-a)^2 - \frac{1}{2}\int_a^b\int_a^y h(x,y)K_1(f,g)\,dxdy,$$

$$(2.2)\quad (3-t)\int_a^b f(y)g(y)(y-a)\,dy \le \int_a^b\Big(\int_a^b\Big(F_0(x,y)(\lambda)g(x)+G_0(x,y)(\lambda)f(x)\Big)dx\Big)dy + \frac{1}{2}(1-t)g(b)f(b)(b-a)^2 - \frac{1}{2}\int_a^b\int_y^b h(x,y)K_1(f,g)\,dxdy,$$

$$(2.3)\quad (3-t)\int_a^b f(y)g(y)(b-a)\,dy \le \int_a^b\Big(\int_a^b\Big(F_0(x,y)(\lambda)g(x)+G_0(x,y)(\lambda)f(x)\Big)dx\Big)dy + \frac{1}{2}(1-t)(b-a)^2\big(f(a)g(a)-g(b)f(b)\big) - \frac{1}{2}\int_a^b\int_y^b h(x,y)K_1(f,g)\,dxdy,$$

where

$$h(x,y)=(x-y-2tx+2ty),$$
$$K_1(f,g) = \Big(f'\big(tx+(1-t)y\big)g(x)+g'\big(tx+(1-t)y\big)f(x)\Big).$$

Proof. The assumptions that f and g are nonnegative and convex, from the Lemma 2.1, imply that we may assume that f, $g\in C^1$ and that we have the following estimates

$$(2.4)\quad f\big(\lambda[tx+(1-t)y]+(1-\lambda)[(1-t)x+ty]\big) \ge f(x)+(1-t)(y-x)f'(x) + (1-\lambda)h(x,y)f'\big(tx+(1-t)y\big),$$

$$(2.5)\quad f\big((1-\lambda)[tx+(1-t)y]+\lambda[(1-t)x+ty]\big) \ge f(x)+(1-t)(y-x)f'(x) + \lambda h(x,y)f'\big(tx+(1-t)y\big),$$

$$(2.6)\quad g\big(\lambda[tx+(1-t)y]+(1-\lambda)[(1-t)x+ty]\big) \ge g(x)+(1-t)(y-x)g'(x) + (1-\lambda)h(x,y)g'\big(tx+(1-t)y\big),$$

$$(2.7)\quad g\big((1-\lambda)[tx+(1-t)y]+\lambda[(1-t)x+ty]\big) \ge g(x)+(1-t)(y-x)g'(x) + \lambda h(x,y)g'\big(tx+(1-t)y\big),$$

for x, $y\in[a,b]$ and t, $\lambda\in[0,1]$ with $h(x,y)=(x-y-2tx+2ty)$. From (2.4), (2.5), (1.3) and (2.6), (2.7), (1.4) it is easy to see that

$$(2.8)\quad F_0(x,y)(\lambda)\ge f(x)+(1-t)(y-x)f'(x) + \frac{1}{2}h(x,y)f'\big(tx+(1-t)y\big),$$

$$(2.9)\quad G_0(x,y)(\lambda)\ge g(x)+(1-t)(y-x)g'(x) + \frac{1}{2}h(x,y)g'\big(tx+(1-t)y\big),$$

for x, $y\in[a,b]$ and t, $\lambda\in[0,1]$. Multiplying (2.8) by $g(x)$ and (2.9) by $f(x)$ and then adding, we obtain

$$(2.10)\quad F_0(x,y)(\lambda)g(x)+G_0(x,y)(\lambda)f(x) \ge 2f(x)g(x)+(1-t)(y-x)\frac{d}{dx}\Big(f(x)g(x)\Big) + \frac{1}{2}h(x,y)K_1(f,g),$$

where

$$K_1(f,g) = \Big(f'\big(tx+(1-t)y\big)g(x)+g'\big(tx+(1-t)y\big)f(x)\Big).$$

Integrating the inequality (2.10) over x from a to y we have

$$(2.11)\quad \int_a^y\Big(F_0(x,y)(\lambda)g(x)+G_0(x,y)(\lambda)f(x)\Big)dx \ge (3-t)\int_a^y f(x)g(x)\,dx-(1-t)(y-a)f(a)g(a) + \frac{1}{2}\int_a^y(x-y-2tx+2ty)K_1(f,g)\,dx.$$

Further, integrating both sides of inequality (2.11) with respect to y from a to b we get

$$\int_a^b\int_a^y\Big(F_0(x,y)(\lambda)g(x)+G_0(x,y)(\lambda)f(x)\Big)dxdy \ge (3-t)\int_a^b(b-y)f(y)g(y)\,dy - (1-t)(y-a)f(a)g(a)(b-a)^2 + \frac{1}{2}\int_a^b\int_a^y(x-y-2tx+2ty)K_1(f,g)\,dxdy,$$

where

$$K_1(f,g) = \Big(f'\big(tx+(1-t)y\big)g(x)+g'\big(tx+(1-t)y\big)f(x)\Big).$$

Rewriting of last inequality we get the required inequality in (2.1). Similarly, by first integrating (2.10) over x from y to b and then integrating the resulting inequality over y from a to b, we get the required inequality in (2.2). The inequality (2.3) is obtained by adding the inequalities (2.1) and (2.2). The proof is complete. □

Setting $t \equiv 1$ in Theorem 2.2, then, we have the inequalities in [6, Theorem 1].

3. Further results. Our next result deals with the slight variants of the inequalities given in Theorem 2.2.

Theorem 3.1. *Let f and g be real-valued, nonnegative and convex function on $[a,b]$ and mappings $F_0(x,y)$ and $G_0(x,y)$ be defined by* (1.3) *and* (1.4). *Then for all t, λ in $[0,1]$ we have*

$$(3.1)\quad \frac{3-t}{2}\int_a^b (b-y)\big(f^2(y)+g^2(y)\big)\,dy \le \int_a^b\bigg(\int_a^y\Big(F_0(x,y)(\lambda)f(x)+G_0(x,y)(\lambda)g(x)\Big)dx\bigg)dy + \frac{1}{4}(1-t)(b-a)^2\big(f^2(a)+g^2(a)\big) - \frac{1}{2}\int_a^b\int_a^y (x-y-2tx+2ty)K_2(f,g)\,dxdy,$$

$$(3.2)\quad \frac{3-t}{2}\int_a^b (y-a)\big(f^2(y)+g^2(y)\big)\,dy \le \int_a^b\bigg(\int_y^b\Big(F_0(x,y)(\lambda)f(x)+G_0(x,y)(\lambda)g(x)\Big)dx\bigg)dy + \frac{1}{4}(1-t)(b-a)^2\big(f^2(b)+g^2(b)\big) - \frac{1}{2}\int_a^b\int_y^b (x-y-2tx+2ty)K_2(f,g)\,dxdy,$$

$$(3.3)\quad \frac{3-t}{2}\int_a^b (b-a)\big(f^2(y)+g^2(y)\big)\,dy \le \int_a^b\bigg(\int_a^b\Big(F_0(x,y)(\lambda)f(x)+G_0(x,y)(\lambda)g(x)\Big)dx\bigg)dy + \frac{1}{4}(1-t)(b-a)^2\big(f^2(a)+f^2(b)+g^2(a)+g^2(b)\big) - \frac{1}{2}\int_a^b\int_a^b (x-y-2tx+2ty)K_2(f,g)\,dxdy,$$

where

$$K_2(f,g) = \Big(f'\big(tx+(1-t)y\big)f(x)+g'\big(tx+(1-t)y\big)g(x)\Big).$$

Proof. As in the proof of Theorem 2.2, from the assumptions we have the estimates (2.8) and (2.9). Multiplying (2.8) by $f(x)$ and (2.9) by $g(x)$ and then adding, we obtain

$$(3.4)\quad F_0(x,y)(\lambda)f(x)+G_0(x,y)(\lambda)g(x) \ge f^2(x)+g^2(x) + (1-t)(y-x)\big(f(x)f'(x)+g(x)g'(x)\big) + \frac{1}{2}(x-y-2tx+2ty)K_2(f,g),$$

where

$$K_2(f,g) = \Big(f'\big(tx+(1-t)y\big)f(x)+g'\big(tx+(1-t)y\big)g(x)\Big).$$

Integrating the inequality (3.4) over x from a to y we have

$$(3.5)\quad \int_a^y\Big(F_0(x,y)(\lambda)f(x)+G_0(x,y)(\lambda)g(x)\Big)\,dx \ge \frac{3-t}{2}\int_a^y\big(f^2(x)+g^2(x)\big)\,dx - \frac{1-t}{2}(y-a)\big(f^2(a)+g^2(a)\big) + \frac{1}{2}\int_a^y (x-y-2tx+2ty)K_2(f,g)\,dx.$$

Further, integrating both sides of (3.5) with respect to y from a to b we get

$$\int_a^b\int_a^y\big(F_0(x,y)(\lambda)f(x)+G_0(x,y)(\lambda)g(x)\big)\,dxdy \ge \frac{3-t}{2}\int_a^b (b-y)\big(f^2(y)+g^2(y)\big)\,dy - \frac{(1-t)(b-a)^2}{4}\big(f^2(a)+g^2(a)\big) + \frac{1}{2}\int_a^b\int_a^y (x-y-2tx+2ty)K_2(f,g)\,dxdy.$$

Rewriting of last inequality, we get the required inequality in (3.1). The remainder of the proof follows by the same arguments as mentioned in the proof of Theorem 2.2 with suitable modifications and hence the proof is complete. □

In next theorem we shall give some inequalities that are analogous to given in Theorem 2.2 involving only one convex function.

Theorem 3.2. *Let f and g be real-valued, nonnegative and convex function on $[a,b]$ and map-*

pings $F_0(x,y)$ and $G_0(x,y)$ be defined by (1.3) *and* (1.4)*. Then for all t, λ in $[0,1]$ we have*

$$(2-t)\int_a^b f(y)(b-y)\,dy \le \int_a^b \left[\int_a^y \left(\int_0^1 f(x,y;t,\lambda)\,d\lambda\right)dx\right]dy + \frac{(1-t)(b-a)^2}{2}f(a) - \frac{1}{2}\int_a^b\int_a^y (x-y-2tx+2ty) \times f'(tx+(1-t)y)\,dxdy, \tag{3.6}$$

$$(2-t)\int_a^b f(y)(y-a)\,dy \le \int_a^b \left[\int_y^b \left(\int_0^1 f(x,y;t,\lambda)\,d\lambda\right)dx\right]dy + \frac{(1-t)(b-a)^2}{2}f(b) - \frac{1}{2}\int_a^b\int_y^b (x-y-2tx+2ty) \times f'(tx+(1-t)y)\,dxdy, \tag{3.7}$$

$$(2-t)\int_a^b f(y)(b-a)\,dy \le \int_a^b \left[\int_a^b \left(\int_0^1 f(x,y;t,\lambda)\,d\lambda\right)dx\right]dy + \frac{(1-t)(b-a)^2}{2}\big(f(a)+f(b)\big) - \frac{1}{2}\int_a^b\int_a^b (x-y-2tx+2ty) \times f'(tx+(1-t)y)\,dxdy, \tag{3.8}$$

where

$$f(x,y;t,\lambda) = f\big(\lambda[tx+(1-t)y]+(1-\lambda)[(1-t)x+ty]\big).$$

Proof. To prove the inequality (3.6), as in the proof of Theorem 2.2 from assumptions we have the estimate (2.1). Integrating both sides of (2.1) over t from 0 to 1 we have

$$\int_0^1 f\big(\lambda[tx+(1-t)y] + (1-\lambda)[(1-t)x+ty]\big)\,d\lambda \le f(x)+(1-t)(y-x)f'(x) + \frac{1}{2}h(x,y)f'\big(tx+(1-t)y\big), \tag{3.9}$$

where $h(x,y)=(x-y-2tx+2ty)$.

Now first integrating both sides of (3.9) over x from a to y and after that integrating the resulting inequality over y from a to b we get the required inequality in (3.6). Similarly, by first integrating (3.9) over x from y to b and then integrating the resulting inequality over y from a to b we get the required inequality in (3.7). The inequality (3.8) is obtained by adding the inequalities (3.6) and (3.7). The proof of Theorem 3.2 is thus completed. □

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