Prime geodesic theorem via the explicit formula of Ψ for hyperbolic 3-manifolds

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Abstract: We obtain a lower bound for the error term of the prime geodesic theorem for hyperbolic 3-manifolds. Our result is $\Omega_\pm(x(\log\log x)^{1/3}/\log x)$. We also generalize Sarnak's upper bound $O(x^{(5/3)+\varepsilon})$ to compact manifolds.

Key words: Lower bound; prime geodesic theorem; explicit formula.

1. Introduction. To estimate the error term of the prime geodesic theorem is one of central subjects in the spectral theory of hyperbolic manifolds. For a $(d+1)$-dimensional hyperbolic manifold with Γ being the fundamental group, the prime geodesic theorem is

$$\pi_\Gamma(x) = \mathrm{li}(x^d) + \sum_{n=1}^{M} \mathrm{li}(x^{s_n}) + (\text{error}), \tag{1.1}$$

where $\pi_\Gamma(x)$ is the number of prime geodesics P whose length $l(P)$ satisfies that $N(P) := e^{l(P)} \le x$, and $s_1, \ldots, s_M$ are the zeros of the Selberg zeta function $Z(s)$ in the interval $(d/2, d)$.

In this paper, we give estimates of the error term in (1.1) for 3-dimensional hyperbolic manifolds. The main theorem is as follows:

Theorem 1.1. *When $\Gamma \subset PSL(2,\mathbf{C})$ is a cocompact subgroup, or a cofinite subgroup satisfying that $\sum_{\gamma_n>0} x^{\beta_n-1}/{\gamma_n}^2 = O(1/(1+(\log x)^3))$ where $\beta_n + i\gamma_n$ are poles of the scattering determinant,*

$$\pi_\Gamma(x) = \mathrm{li}(x^2) + \sum_{n=1}^{M} \mathrm{li}(x^{s_n}) + \Omega_\pm\left(\frac{x(\log\log x)^{1/3}}{\log x}\right) \quad \text{as } x \to \infty.$$

The conjectural exponent of x in the error term in (1.1) is $d/2$. Theorem 1.1 gives a sharp estimate in that sense.

Remark 1. Theorem 1.1 is a generalization of the result in Hejhal [5], in which he obtained a lower bound in 2-dimensional cases i.e. $d = 1$.

Remark 2. The assumption in Theorem 1.1 for noncocompact cases is satisfied by Bianchi groups Γ associated to imaginary quadratic number fields $K = \mathbf{Q}(\sqrt{-D})$, i.e.

$$\Gamma = \Gamma_D = PSL(2, O_K) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a,b,c,d \in O_K, ad-bc = 1 \right\} \Big/ \{\pm 1\},$$

where O_K is the ring of integers of K.

2. Selberg zeta function. We first introduce the Selberg zeta function of Γ for a 3-dimensional hyperbolic manifold.

Throughout this paper we put G to be $PSL(2,\mathbf{C})$ and Γ to be a cofinite subgroup of G. Let j be an element in the quaternion field which satisfies $j^2 = -1$, $ij = -ji$, and let $\mathbf{H}$ be the 3-dimensional hyperbolic space:

$$\mathbf{H} := \{ v = z + yj \mid z = x_1 + x_2 i \in \mathbf{C}, y > 0\}$$

with the Riemannian metric

$$dv^2 = \frac{{dx_1}^2 + {dx_2}^2 + dy^2}{y^2}.$$

It induces a hyperbolic distance $d(v, v')$ given by

$$\cosh d(v,v') = \frac{|z-z'|^2 + y^2 + {y'}^2}{2yy'},$$

where $v = z + yj$ and $v' = z' + y'j$. The volume measure is given by

$$\frac{dx_1 dx_2 dy}{y^3}.$$

The group $PSL(2,\mathbf{C})$ acts on $\mathbf{H}$ transitively by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(v) := (av+b)(cv+d)^{-1} = \frac{(az+b)\overline{(cz+d)} + a\bar{c}y^2 + yj}{|cz+d|^2 + |c|^2y^2}.$$

The Laplacian for $\mathbf{H}$ is defined by

$$\Delta := -y^2\left(\frac{\partial^2}{\partial {x_1}^2} + \frac{\partial^2}{\partial {x_2}^2} + \frac{\partial^2}{\partial y^2}\right) + y\frac{\partial}{\partial y}.$$

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We denote the eigenvalues of Δ by $0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_M \le 1 < \lambda_{M+1} \cdots$.

For conjugacy classes we give the classification.

Definition 2.1. An element $P \in \Gamma - \{1\}$ is called

parabolic iff $|\operatorname{tr}(P)| = 2$ and $\operatorname{tr}(P) \in \mathbf{R}$,
hyperbolic iff $|\operatorname{tr}(P)| > 2$ and $\operatorname{tr}(P) \in \mathbf{R}$,
elliptic iff $|\operatorname{tr}(P)| < 2$ and $\operatorname{tr}(P) \in \mathbf{R}$,

and loxodromic in all other cases. An element of $PSL(2, \mathbf{C})$ is called parabolic, elliptic, hyperbolic, loxodromic if its preimages in $SL(2, \mathbf{C})$ have this property. A conjugacy class $\{P\}$ in Γ is called hyperbolic, elliptic, parabolic if each P in the class has this property.

The norm of a hyperbolic or loxodromic element P is defined by $N(P) = |a(P)|^2$, if $a(P) \in \mathbf{C}$ is the eigenvalue of $P \in G$ such that $|a(P)| > 1$.

Definition 2.2. An element $P \in \Gamma - \{1\}$ is called primitive iff it is not an essential power of any other element. A conjugacy class $\{P\}$ in Γ is called primitive if each P in the class has this property.

For every hyperbolic matrix $P \in \Gamma$ there exist exactly one primitive hyperbolic element $P_0 \in \Gamma$ and exactly one $n \in \mathbf{N}$ such that $P = {P_0}^n$. We recognize that $\pi_\Gamma(x)$ in Section 1 is the number of P_0 which is primitive hyperbolic or loxodromic and satisfies $N(P_0) \le x$.

Definition 2.3. For $\operatorname{Re}(s) > 2$, the Selberg zeta function for Γ is defined by

$$Z(s) := \prod_{\{P_0\}} \prod_{(k,l)} (1 - a(P_0)^{-2k} \overline{a(P_0)}^{-2l} N(P_0)^{-s}),$$

where the product on $\{P_0\}$ is taken over all primitive hyperbolic or loxodromic conjugacy classes of Γ, and (k, l) runs through all the pairs of positive integers satisfying the following congruence relation: $k \equiv l \pmod{m(P_0)}$ with $m(P)$ the order of the torsion of the centralizer of P.

For the Selberg zeta function, Elstrodt, Grunewald and Mennicke proved the following Lemma.

Lemma 2.4 [2, p. 208, Lemma 4.2]. *For* $\operatorname{Re}(s) > 2$, *we have*

$$\frac{Z'}{Z}(s) = \sum_{\{P\}} \frac{N(P) \log N(P_0)}{m(P)|a(P) - a(P)^{-1}|^2} N(P)^{-s},$$

where $\{P\}$ *runs through the hyperbolic or loxodromic conjugacy classes of* Γ *and* P_0 *is a primitive element associated with* P.

By comparing with the von-Mangoldt function in the logarithmic derivative of the Riemann zeta function, the following definition is natural.

Definition 2.5. For a hyperbolic or loxodromic element P of Γ, we define

$$\Lambda_\Gamma(P) := \frac{N(P) \log N(P_0)}{m(P)|a(P) - a(P)^{-1}|^2},$$

and

$$\Psi_\Gamma(x) := \sum_{\substack{\{P\} \\ N(P) \le x}} \Lambda_\Gamma(P),$$

where $\{P\}$ runs through hyperbolic or loxodromic classes of Γ and P_0 is a primitive element associated with P.

Then we have

$$\frac{Z'}{Z}(s) = \sum_{\{P\}} \Lambda_\Gamma(P) N(P)^{-s}.$$

Theorem 1.1 can be shown by using the explicit formula for $\Psi_2(x) := \int_1^x \Psi_1(t)dt$, where $\Psi_1(x) := \int_1^x \Psi_\Gamma(t)dt$. Though Hejhal [5] used the explicit formula for $\Psi_1(x)$, in our case the order of $Z(s)$ is three and abundance of the zeros of $Z(s)$ gives rise to a difficulty concerning the estimate of $\Psi_1(x)$. We overcame it by considering $\Psi_2(x)$.

3. Outline of proofs. In noncocompact but cofinite cases, we have to consider the contribution from parabolic classes and continuous spectra. Since we can omit the contribution of the continuous spectra under the assumption in Theorem 1.1, it suffices to give proofs for cocompact cases.

We introduce the following property of $Z(s)$ for cocompact Γ. Let $s_n = 1 + it_n$ and $\tilde{s}_n = 1 - it_n$ be the zeros of $Z(s)$, where $t_n := \sqrt{\lambda_n - 1}$.

Proposition 3.1. *We have*

$$\frac{Z'}{Z}(s) = \frac{1}{s-2} + \sum_{|s-s_n|<1} \frac{1}{s - s_n} + \sum_{|s-\tilde{s}_n|<1} \frac{1}{s - \tilde{s}_n} + O(|s|^2 + 1),$$

where the sums are taken over s_n *and* $\tilde{s}_n$ *with* $\operatorname{Re}(s_n) = 1$.

This proposition can be deduced from the determinant expression of $Z(s)$ in [7, p. 766, Theorem 4.4] and the functional equation in [2, p. 209, Corollary 4.4].

About the distribution of the imaginary parts of the complex zeros of $Z(s)$ for cocompact Γ, we have the following proposition.

Proposition 3.2 [2, p. 215, Theorem 5.6]. *Suppose that $T > 0$, $T \neq t_n$, then for all $n \geq M+1$, the counting function $N(T) := \sharp\{n \mid n \geq M+1,\ t_n < T\}$ satisfies*

$$N(T) = \frac{\mathrm{Vol}(\Gamma\backslash\mathbf{H})}{6\pi^2}T^3 + O(T^2) \quad as \quad T \to \infty,$$

where $\mathrm{Vol}(\Gamma\backslash\mathbf{H})$ *is the volume of the fundamental domain* $\Gamma\backslash\mathbf{H}$.

Since we can express $\Psi_2(x)$ as

$$\Psi_2(x) = \frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i} \frac{x^{s+2}}{s(s+1)(s+2)}\frac{Z'}{Z}(s)ds,$$

by the standard method [6, p. 31, Theorem B], we have the following theorem by Propositions 3.1 and 3.2.

Theorem 3.3. *Let* $\Psi_1(x) := \int_1^x \Psi_\Gamma(t)dt$. *Then we have*

$$\begin{aligned}\Psi_1(x) &= \alpha x + \beta x\log x + \alpha_1 \\ &\quad + \sum_{n=0}^{M}\frac{x^{s_n+1}}{s_n(s_n+1)} + \sum_{n=0}^{M}\frac{x^{\tilde{s}_n+1}}{\tilde{s}_n(\tilde{s}_n+1)} \\ &\quad + \sum_{t_n\geq 0}\frac{x^{s_n+1}}{s_n(s_n+1)} + \sum_{t_n>0}\frac{x^{\tilde{s}_n+1}}{\tilde{s}_n(\tilde{s}_n+1)}\end{aligned}$$

with some constants α, β and α_1, where $s_n = 1+it_n$ and $\tilde{s}_n = 1-it_n$ are the zeros of $Z(s)$.

Under the notation in Theorem 3.3, we have a relation between $\Psi_\Gamma(x)$ and $\pi_\Gamma(x)$:

$$\begin{aligned}&\pi_\Gamma(x) - \left(\sum_{n=0}^{M}\mathrm{li}(x^{s_n}) + \sum_{n=0}^{M}\mathrm{li}(x^{\tilde{s}_n})\right) \\ &= \frac{1}{\log x}\left\{\Psi_\Gamma(x) - \left(\alpha + \beta\log x + \beta + \sum_{n=0}^{M}\frac{x^{s_n}}{s_n}\right.\right. \\ &\qquad\left.\left. + \sum_{n=0}^{M}\frac{x^{\tilde{s}_n}}{\tilde{s}_n}\right)\right\} + O\left(\frac{x}{\log x}\right).\end{aligned}$$

We reach Theorem 1.1 by estimating $\Psi_\Gamma(x)$:

$$\begin{aligned}\Psi_\Gamma(x) &= \alpha + \beta\log x + \beta + \sum_{n=0}^{M}\frac{x^{s_n}}{s_n} \\ &\quad + \sum_{n=0}^{M}\frac{x^{\tilde{s}_n}}{\tilde{s}_n} + \Omega_\pm\left(x(\log\log x)^{1/3}\right).\end{aligned}$$

4. *O*-result. The upper estimates of the error term in (1.1) have been studied by many people in the case of $d = 1$. For higher dimensional cases, the only known result is Sarnak's error term $O(x^{(5/3)+\varepsilon})$ in [10] for $\Gamma = PSL(2, O_K)$ with K being an imaginary quadratic field ($\neq \mathbf{Q}(i), \mathbf{Q}(\sqrt{-3})$) of class number one. (Some conditional results are obtained in [8].) We generalize Sarnak's estimate to cocompact groups and to general Bianchi groups by using the 'explicit formula' of $\Psi_\Gamma(x)$.

The explicit formula is as follows:

Theorem 4.1. *Suppose that Γ is a cocompact group or a Bianchi group. Let $1 \leq T < x^{1/2}$. Then we have*

$$\begin{aligned}\Psi_\Gamma(x) &= \frac{1}{2}x^2 + \sum_{n=0}^{M}\frac{1}{s_n}x^{s_n} + \sum_{n=0}^{M}\frac{1}{\tilde{s}_n}x^{\tilde{s}_n} \\ &\quad + \sum_{0<t_n\leq T}\frac{1}{s_n}x^{s_n} + \sum_{0<t_n\leq T}\frac{1}{\tilde{s}_n}x^{\tilde{s}_n} + O\left(\frac{x^2}{T}\log x\right)\end{aligned}$$

as $x \to \infty$, where $s_n = 1+it_n$ and $\tilde{s}_n = 1-it_n$ are the zeros of $Z(s)$ coming from discrete spectra.

Taking $T = x^{1/3}$ gives

$$\begin{aligned}\Psi_\Gamma(x) &= \frac{1}{2}x^2 + \sum_{n=0}^{M}\frac{1}{s_n}x^{s_n} + \sum_{n=0}^{M}\frac{1}{\tilde{s}_n}x^{\tilde{s}_n} \\ &\quad + \sum_{0<t_n\leq T}\frac{1}{s_n}x^{s_n} + \sum_{0<t_n\leq T}\frac{1}{\tilde{s}_n}x^{\tilde{s}_n} + O(x^{(5/3)+\varepsilon}).\end{aligned}$$

From the relation

$$\begin{aligned}\pi_\Gamma(x) - \mathrm{li}(x^2) &= \int_2^x \frac{\Psi_\Gamma(u)du}{u\log^2 u} + \frac{2\Psi_\Gamma(x) - x^2}{2\log x} \\ &\quad - \int_2^{x^2}\frac{du}{\log^2 u} + O(x),\end{aligned}$$

we have the following theorem:

Theorem 4.2. *When $\Gamma \subset PSL(2, \mathbf{C})$ is a cocompact subgroup or $\Gamma = PSL(2, O_K)$ with K an imaginary quadratic field,*

$$\pi_\Gamma(x) = \mathrm{li}(x^2) + \sum_{n=1}^{M}\mathrm{li}(x^{s_n}) + O(x^{(5/3)+\varepsilon})$$

as $x \to \infty$.

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