A note on the mean value of the zeta and L-functions. X

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Abstract: The present note reports on an explicit spectral formula for the fourth moment of the Dedekind zeta function ζ_{F} of the Gaussian number field $\mathrm{F} = \mathbf{Q}(i)$, and on a new version of the sum formula of Kuznetsov type for $\mathrm{PSL}_2(\mathbf{Z}[i])\backslash\mathrm{PSL}_2(\mathbf{C})$. Our explicit formula (Theorem 5, below) for ζ_{F} gives rise to a solution to a problem that has been posed on p. 183 of [M3] and, more explicitly, in [M4]. Also, our sum formula (Theorem 4, below) is an answer to a problem raised in [M4] concerning the inversion of a spectral sum formula over the Picard group $\mathrm{PSL}_2(\mathbf{Z}[i])$ acting on the three dimensional hyperbolic space (the K-trivial situation). To solve this problem, it was necessary to include the K-nontrivial situation into consideration, which is analogous to what has been experienced in the modular case.

Key words: Zeta-function; imaginary quadratic number field; Kloosterman sum; sum formula; automorphic representation; spectral decomposition.

1. Introduction. We are concerned with the fourth moment

$$\mathcal{Z}_2(g,\mathrm{F}) = \int_{-\infty}^{\infty} \left|\zeta_{\mathrm{F}}\left(\frac{1}{2}+it\right)\right|^4 g(t)dt,$$

with g a holomorphic function having rapid decay in any fixed horizontal strip. See [M4] for the motivation behind this problem, in the light of the theory of the Riemann zeta-function. Our explicit spectral decomposition for $\mathcal{Z}_2(g,\mathrm{F})$ is analogous to that in Theorem 4.2 in [M3], which treats the fourth moment of the Riemann zeta function. As in *loc. cit.*, the proof is based on an application of a sum formula relating the spectral decomposition of

$$L^2(\mathrm{PSL}_2(\mathbf{Z}[i])\backslash \mathrm{PSL}_2(\mathbf{C}))$$

to a sum of Kloosterman sums over F. For the present purpose, existing versions of the sum formula, as in Theorem 2.2 in [MW], cannot be used directly. One cannot restrict oneself to K-trivial vectors as is done in [MW], and also needs specific information concerning the Bessel transformation in the sum formula. Theorem 1 below gives the extension of a Kuznetsov type formula to Fourier coefficients of automorphic forms with arbitrary K-type. Full proofs are given in [BM], which is expected to appear elsewhere. Moreover, a relevant Bessel transform (the B, below) is thoroughly analyzed, and a partial inversion of it is proved (Theorem 2, below), which implies a solution to the inversion problem mentioned in the abstract.

The restriction to the Gaussian number field is by no means essential, as far as the sum formula for Kloosterman sums is concerned. In treating Kloosterman sums over an arbitrary imaginary quadratic number field $\mathbf{Q}(\sqrt{-D})$ ($\mathbf{Z} \ni D > 0$), we deal with the space

$$L^2(\mathrm{PSL}_2(\mathcal{O})\backslash \mathrm{PSL}_2(\mathbf{C})),$$

where $\mathrm{PSL}_2(\mathcal{O})$ is the Bianchi group, with $\mathcal{O}$ the ring of integers in $\mathbf{Q}(\sqrt{-D})$. The difference in the argument is limited to the discussion about the contribution of Eisenstein series or the non-cuspidal subspace ${}^{e}L^2$ of $L^2(\mathrm{PSL}_2(\mathcal{O})\backslash \mathrm{PSL}_2(\mathbf{C}))$ that are generated by them. The Bianchi group has, in general, inequivalent cusps, the number of which is equal to the class number of the field, and as much the contribution of ${}^{e}L^2$ splits into classes. But each of those contributions is similar, in structure, to that of the sole cusp at infinity present in our sum formula (Theorem 4, below). Further, congruence subgroups of Bianchi groups can also be included into our discussion, solely at the cost of greater complexity.

In contrast to this, we exploit the fact that F has class number one, in deriving the explicit spec-

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tral decomposition for $\mathcal{Z}_2(g,\mathrm{F})$. Our argument extends readily to $\mathbf{Q}(\sqrt{-D})$, provided the field has class number one. But in the general situation there exists a difficulty. This stems from the nature of a splitting argument on which depends the reduction of $\mathcal{Z}_2(g,\mathrm{F})$, or rather a closely related quantity, into a sum of Kloosterman sums. We do not know whether this splitting argument can be extended to the case of class number larger than one.

2. Spectral sum formula. Denote $G = \mathrm{PSL}_2(\mathbf{C})$, and $\Gamma = \mathrm{PSL}_2(\mathbf{Z}[i])$. The Hilbert space $L^2(\Gamma\backslash G)$ has an orthogonal decomposition

$$\mathbf{C} \oplus {}^0L^2(\Gamma\backslash G) \oplus {}^eL^2(\Gamma\backslash G),$$

with successively the constant functions, the cuspidal subspace, and the subspace spanned by integrals of Eisenstein series.

The cuspidal subspace is the closure of the direct sum $\bigoplus V$ of an orthogonal system of irreducible subspaces V, which can be chosen to be invariant under all Hecke operators. Under suitable normalizations, the elements of V have a Fourier expansion with coefficients $c_V(n) = c_V(1)t_V(n)$, $n \in \mathbf{Z}[i]$, where $t_V(n) \in \mathbf{R}$ is the eigenvalue of the corresponding Hecke operator. The type of the representation V is characterized by a discrete set of parameter pairs $(\nu_V, p_V) \in i\mathbf{R}\times\mathbf{Z}$, such that $(1/8)((\nu_V \mp p_V)^2 - 1)$ are the eigenvalues in V of the two Casimir elements that generate the center of the universal enveloping algebra of G.

For the Eisenstein series, the corresponding parameter (ν, p) runs continuously through $i\mathbf{R}\times\mathbf{Z}$. The Fourier coefficients can be expressed in terms of the Grössencharakter zeta function

$$\zeta_{\mathrm{F}}(s,p) = \frac{1}{4}\sum_{n\neq 0}\left(\frac{n}{|n|}\right)^{4p}|n|^{-2s}$$

and divisor sums

$$\sigma_\nu(n,p) = \frac{1}{4}\sum_{d|n}\left(\frac{d}{|d|}\right)^{4p}|d|^{2\nu}, \quad d,n\in\mathbf{Z}[i].$$

These spectral data are related by the sum formula to Kloosterman sums

$$S_{\mathrm{F}}(m,n;c) = \sum_{\substack{d \bmod c\\ (d,c)=1}} \exp\left(2\pi i \,\mathrm{Re}\left(\frac{md+n\tilde d}{c}\right)\right),$$

$c,d,m,n\in\mathbf{Z}[i]$, $d\tilde d \equiv 1 \bmod c$.

To formulate that relation, an integral transformation is used with the following kernel function:

$$\mathcal{K}_{\nu,p}(u) = \frac{1}{\sin\pi\nu}\{\mathcal{J}_{-\nu,-p}(u) - \mathcal{J}_{\nu,p}(u)\},$$

$$\mathcal{J}_{\nu,p}(u) = \left|\frac{u}{2}\right|^{2\nu}\left(\frac{u}{|u|}\right)^{-2p} \cdot \sum_{m,n\geq 0}\frac{(-1)^{n+m}(u/2)^{2n}(\bar u/2)^{2m}}{n!m!\Gamma(\nu-p+n+1)\Gamma(\nu+p+m+1)}.$$

So $\mathcal{J}_{\nu,p}(u) = J_{\nu-p}(u)J_{\nu+p}(\bar u)$, with an appropriate choice of the arguments of u and $\bar u$.

Theorem 1. *Let $h(\nu,p)$ be a function defined over the set $i\mathbf{R}\times\mathbf{Z}$, satisfying the conditions*:

1. $h(\nu,p) = h(-\nu,-p)$,
2. *$h(\nu,p)$ is regular for $|\mathrm{Re}\,\nu| \leq (1/2)+a$ with a small $a>0$,*
3. *$h(\nu,p) \ll (1+|\nu|+|p|)^{-4-b}$ with a small $b>0$.*

Then we have, for any non-zero $m,n\in\mathbf{Z}[i]$,

$$\sum_V |c_V(1)|^2 t_V(m)t_V(n)h(\nu_V,p_V) + \frac{1}{2\pi i}\sum_{2\mathbf{Z}\ni p}\left(\frac{mn}{|mn|}\right)^p \cdot\int_{(0)}\frac{\sigma_\nu(m,-p/2)\sigma_\nu(n,-p/2)}{|mn|^\nu|\zeta_{\mathrm{F}}(1+\nu,p/2)|^2}h(\nu,p)d\nu$$
$$= \frac{\delta_{m,n}+\delta_{m,-n}}{4\pi^3 i}\sum_{\mathbf{Z}\ni p}\int_{(0)}h(\nu,p)(p^2-\nu^2)d\nu + \sum_{\mathbf{Z}[i]\ni c\neq 0}\frac{1}{|c|^2}S_{\mathrm{F}}(m,n;c)\mathrm{B}h\left(\frac{2\pi}{c}\sqrt{mn}\right),$$

with (0) being the imaginary axis. Here

$$\mathrm{B}h(u) = \sum_{\mathbf{Z}\ni p}\frac{1}{8\pi i}\int_{(0)}\mathcal{K}_{\nu,p}(u)h(\nu,p)(p^2-\nu^2)d\nu.$$

Convergence of these expressions is absolute throughout.

Basic to the proof is the computation of the scalar product of two Poincaré series in two ways: A spectral computation, corresponding to the decomposition of $L^2(\Gamma\backslash G)$, and a geometric computation, by taking apart one of the Poincaré series. The general approach is an elaboration of the method in [MW], but some complications occur at the step from $L^2(\Gamma\backslash G/\mathrm{SU}(2))$ to $L^2(\Gamma\backslash G)$.

The absence of exceptional eigenvalues for the present choice of Γ, the fact that F has class number one, and the Weil bound for Kloosterman sums, simplify the proof. The extension to general imaginary quadratic number fields and congruence subgroups is possible, as is mentioned above.

3. Geometric sum formula. Theorem 1 has the independent test function on the spectral side. To use the sum formula for the present purpose, and in other applications, it is useful to have a partial inversion of the Bessel transform B:

Theorem 2. *We put*

$$\mathrm{K}f(\nu,p)=\int_{\mathbf{C}^*}\mathcal{K}_{\nu,p}(u)f(u)\frac{d\,\mathrm{Re}\,u d\,\mathrm{Im}\,u}{|u|^2}.$$

Then, for any f that is even, smooth and compactly supported on $\mathbf{C}^*=\mathbf{C}\backslash\{0\}$, we have

$$2\pi\mathrm{BK}f=f.$$

Practical bounds for the Bessel kernel $\mathcal{K}_{\nu,p}$ can be derived from the following integral representation:

Theorem 3. *Let $|\,\mathrm{Re}\,\nu|<1/4$. Then we have, for any $p\in\mathbf{Z}$ and non-zero $u\in\mathbf{C}$,*

$$\mathcal{K}_{\nu,p}(u)=(-1)^p\frac{2}{\pi}\int_0^\infty y^{2\nu-1}\left(\frac{ye^{i\vartheta}+(ye^{i\vartheta})^{-1}}{|ye^{i\vartheta}+(ye^{i\vartheta})^{-1}|}\right)^{2p} \times J_{2p}\left(|u||ye^{i\vartheta}+(ye^{i\vartheta})^{-1}|\right)dy,$$

where $u=|u|e^{i\vartheta}$.

With these results in hand, the spectral sum formula can be applied to $h=\mathrm{K}f$, with f smooth and compactly supported on $\mathbf{C}^*$. For such test functions, the first term in the right hand side of the spectral sum formula vanishes. Extending the class of test functions leads to the *geometric sum formula*:

Theorem 4. *Let f be an even function on $\mathbf{C}^*$. Let us suppose that there exist constants ρ and σ such that $0<\rho<1/2<\sigma$, and*

1. *$f(u)=O(|u|^{2\sigma})$ as $|u|\downarrow 0$,*
2. *f is six times continuously differentiable, and for $a+b\le 6$*

$$\int_{\mathbf{C}^*}|(u\partial_u)^a f(u)|^2|u|^{b-2\rho}\frac{d\,\mathrm{Re}\,u d\,\mathrm{Im}\,u}{|u|^2}<\infty.$$

Then we have, for any non-zero $m,n\in\mathbf{Z}[i]$,

$$\sum_{\mathbf{Z}[i]\ni c\neq 0}\frac{1}{|c|^2}S_{\mathrm{F}}(m,n;c)f\left(\frac{2\pi}{c}\sqrt{mn}\right)$$
$$=2\pi\sum_V|c_V(1)|^2t_V(m)t_V(n)\mathrm{K}f(\nu_V,p_V)$$
$$-i\sum_{2\mathbf{Z}\ni p}\left(\frac{mn}{|mn|}\right)^p$$
$$\cdot\int_{(0)}\frac{\sigma_\nu(m,-p/2)\sigma_\nu(n,-p/2)}{|mn|^\nu|\zeta_{\mathrm{F}}(1+\nu,p/2)|^2}\mathrm{K}f(\nu,p)d\nu.$$

The convergence is absolute throughout.

Extension to general imaginary quadratic number fields and congruence subgroups is possible.

4. Explicit formula. Once the geometric sum formula is available, the approach in Chapter 4 of [M3] can be carried out. Here the fact that F has class number one is heavily used, as has been remarked above.

The quantity $\mathcal{Z}_2(g,\mathrm{F})$ equals $\mathcal{J}(1/2,1/2,1/2,1/2;g)$ plus a linear combination of the values of g and g' at $(1/2)i$ and $-(1/2)i$, where $\mathcal{J}$ is the meromorphic function of the z_j that is for $\mathrm{Re}\,z_j>1$ given by

$$\mathcal{J}(z_1,z_2,z_3,z_4;g)=\int_{-\infty}^\infty\zeta_{\mathrm{F}}(z_1+it)\zeta_{\mathrm{F}}(z_2+it) \cdot\zeta_{\mathrm{F}}(z_3-it)\zeta_{\mathrm{F}}(z_4-it)g(t)dt.$$

In the region of absolute convergence, this can be transformed into an infinite sum with terms containing $\hat{g}(2\log|l/k|)$, where $\hat{g}$ is the Fourier transform, and $k,l\in\mathbf{Z}[i]\setminus\{0\}$. Observe that here we use the fact that all ideals in $\mathbf{Z}[i]$ are principal, which is essential in the following application of a splitting argument (see the remark at the end of the introduction above): The sum of the diagonal terms, i.e., those with $k=l$, admits an explicit description in terms of ζ_{F}. The sum of the remaining terms is initially expressed in binary additive divisor sums over $\mathbf{Z}[i]$

$$\mathcal{B}_m(\alpha,\beta;g^*(\cdot;\gamma,\delta))=\sum_{\substack{\mathbf{Z}[i]\ni n\\ n(n+m)\neq 0}}\sigma_\alpha(n)\sigma_\beta(n+m)g^*(n/m;\gamma,\delta),$$

with $m\in\mathbf{Z}[i]\setminus\{0\}$, and

$$g^*(u)=\frac{\hat{g}(2\log|1+1/u|)}{|u|^{2\gamma}|u+1|^{2\delta}}.$$

The complex parameters α, β, γ, δ depend on the z_j. Contrary to the rational case that is treated in [M3], this non-diagonal contribution contains terms with $\hat{g}(0)$ (corresponding to $n=(\epsilon-1)m$, $\epsilon\in\{i,-1,-i\}$).

An appropriate extension of the method in Section 4.3 of [M3] gives a transformation of $\mathcal{B}_m$ into an expression with Kloosterman sums S_{F}. The test function occurs in this re-formulated non-diagonal term by the following integral transformation:

$$\tilde{g}_q(s;\gamma,\delta)=\frac{1}{2\pi}\int_{\mathbf{C}^*}g^*(u;\gamma,\delta) \cdot\left(\frac{u}{|u|}\right)^{-q}|u|^{2s-2}d\,\mathrm{Re}\,u\,d\,\mathrm{Im}\,u,$$

with $q \in \mathbf{Z}$, $\mathrm{Re}(s-\gamma-\delta) < 0$. It has a meromorphic continuation in s.

Application of the geometric sum formula, for all combinations of $m, n \in \mathbf{Z}[i] \setminus \{0\}$, leads to a spectral expression for the non-diagonal term, in which occurs the Hecke series

$$H_V(s) = \frac{1}{4} \sum_{\mathbf{Z}[i] \ni n \neq 0} t_V(n)|n|^{-2s}.$$

The series converges in a right half plane; the function has a holomorphic continuation to $\mathbf{C}$, and a functional equation. As the Hecke series is built with the trivial character, there is no contribution from the irreducible spaces V with $t_V(i) = -1$.

In achieving the spectral decomposition of the contribution of non-diagonal terms, some additional conditions are first imposed on z_j's; that is, Theorem 4 applies well in a rather limited domain of z_j's. This domain does not contain the crucial point $(1/2, 1/2, 1/2, 1/2)$, and hence an analytic continuation is required. This procedure of continuation is carried out along the same lines as in Section 4.6 of [M3]. The final result has the following structure:

Theorem 5. *There are functionals $g \mapsto M_{\mathrm{F}}(g)$ and $g \mapsto \Lambda_{\nu,p}(g)$ on the space of test functions g indicated above such that*

$$\mathcal{Z}_2(g, \mathrm{F}) = M_{\mathrm{F}}(g) + \sum_V |c_V(1)|^2 H_V\left(\frac{1}{2}\right)^3 \Lambda_{\nu_V, p_V}(g)$$
$$+ \frac{1}{2\pi i} \sum_{4\mathbf{Z} \ni p} \int_{(0)} \frac{|\zeta_{\mathrm{F}}((1/2)(1+\nu), (1/4)p)|^6}{|\zeta_{\mathrm{F}}(1+\nu, (1/2)p)|^2} \Lambda_{\nu,p}(g) d\nu.$$

In the above, $\Lambda_{\nu,p}(g)$ is given as the value at $(\alpha, \beta, \gamma, \delta) = (0, 0, 1/2, 1/2)$ of $\Phi_p(\nu; \alpha, \beta, \gamma, \delta; g)$, which is an integral of $\tilde{g}_p(s; \gamma, \delta)$ for suitable values of $\alpha, \beta, \gamma, \delta$, and meromorphically continued. The construction of Φ_p is similar to the corresponding transform in the rational case (see Lemma 4.4 of [M3]). The term $M_{\mathrm{F}}(g)$ contains all other contributions: the diagonal term, the values of g and g' at $\pm(1/2)i$, and the residues picked up in the various steps, which are similar to those terms on p. 175 of [M3].

Our Theorem 5 appears analogous to the corresponding result for the Riemann zeta function, see Theorem 4.2 in [M3]. Especially, the appearance of cubic powers of central values of Hecke series is a common feature. This similarity raises further problems such as seeking for the analogue of the Ω-result obtained in Part VII [IM] concerning the Riemann zeta-function, and the rôle of the Hecke series in the formation of values of the zeta-function ζ_{F} that extends Part IX [M1]. It is hoped that these issues will be discussed in our future works.

Finally it should be remarked that for the mean square of Dedekind zeta-functions of any — real and imaginary — quadratic number fields there exists a relatively satisfactory result [M2]. One may combine the results of [M2] with the argument of Part VIII [M1], on the eighth power moment of the Riemann zeta-function, to deduce a spectral decomposition of

$$\int_{-\infty}^{\infty} \left| \zeta_{\mathbf{Q}(\sqrt{D})}\left(\frac{1}{2} + i(T+t)\right)\right|^4 \exp\left(-\left(\frac{t}{G}\right)^2\right) dt$$

with $D \in \mathbf{Z}$, and $T \geq 0$, $G > 0$ being arbitrary. In this setting the underlying spectral theory is that over $\mathrm{PSL}_2(\mathbf{R})$ modulo the Hecke congruence subgroup of level $|D|$. Thus there are at least two ways to spectrally decompose $\mathcal{Z}_2(g, \mathrm{F})$ with g being the Gaussian distribution. The experience in the rational case suggests, however, that the spectral decomposition stated in Theorem 5 above will turn out to be more fundamental, and practical as well.

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