On an elliptic curve over Q(t) of rank ≥ 9 with a non-trivial 2-torsion point

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Abstract: We show an elliptic curve over $\mathbf{Q}(t)$ of rank at least 9 with a non-trivial 2-torsion point.

Key words: Elliptic curve; rank; 2 torsion point.

In [3], Nagao constructed a family of infinitely many elliptic curves over $\mathbf{Q}$ with a non-trivial rational 2-torsion point and with rank ≥ 6. Also, in [1], Fermigier gave an example of an elliptic curve over $\mathbf{Q}(t)$, of rank at least 8, with a non-trivial 2-torsion point. We showed in [2] that there are infinitely many elliptic curves over $\mathbf{Q}$ with a non-trivial rational 2-torsion point and with rank ≥ 9.

In this note we show an example of elliptic curve with a non-trivial 2-torsion point over $\mathbf{Q}(t)$ of rank at least 9. Let

$$b_1 = u+a, \quad b_2 = u+b, \quad b_3 = u+c,$$
$$b_4 = -u+a, \; b_5 = -u+b, \; b_6 = -u+c$$

and

$$F(x) = \prod_{i=1}^{6}(x^2 - b_i^2),$$

then there are unique polynomials $G(x)$, $r(x) \in K(x)$ where $K = \mathbf{Q}(u,a,b,c)$, with $\deg G(x) = 6$, $\deg r(x) = 4$, and $F(x) = G(x)^2 - r(x)$.

$r(x)$ is of the form,

$$r(x) = Ax^4 + Bx^2 + C \quad \text{where} \quad A, B, C \in K.$$

Now we consider the elliptic curve,

$$\mathcal{E} \qquad y^2 = r(x),$$

There are 6 K-rational points $\mathrm{P}_i(x_i, y_i)$ $(1 \leq i \leq 6)$ on $\mathcal{E}$ where $x_i = b_i$, $v_i = G(b_i)$.

By specializing $a = p^2(p+2q)$, $b = q^2(2p+q)$ and $c = (p+q)^2(p-q)$, we have another point $\mathrm{P}_7(x_7, y_7)$ on $\mathcal{E}$, where

$$x_7 = pq(p+q),$$
$$y_7 = 2pq(-p+q)(p+q)(2p+q)(p+2q)u^2(-p^6 - 3p^5q - 2p^4q^2 + p^3q^3 - 2p^2q^4 - 3pq^5 - q^6 + u^2).$$

We treated the case $p = 2$, $q = 1$ in [2].

Next let

$$l = p^2(-p+q)(2p+q)(2p^2+pq+q^2)$$

and

$$m = -3p^2q^2(p+q)^2,$$

by specializing

$$u = \frac{s^2 - l}{2s},$$

we have a point $\mathrm{P}_8(x_8, y_8)$ on $\mathcal{E}$, where

$$x_8 = \frac{s^2 + l}{2s},$$
$$y_8 = 4p^2q(-p+q)(p+q)(2p+q)(p+2q) \times u(p^2(-3p^2+pq+q^2)(p^2+pq+q^2)^2 + 2q(p+q)u^2).$$

On the other hand, by specializing $u = (w^2 - m)/(2w)$ we have a point $\mathrm{P}_9(x_9, y_9)$ on $\mathcal{E}$, where

$$x_9 = \frac{w^2 + m}{2w},$$
$$y_9 = 4p^2q^2(-p+q)(p+q)^2(2p+q)(p+2q) \times u(-(p^2+pq+q^2)^3 + 2u^2).$$

So we have to solve the Diophantine equation,

$$u = \frac{s^2 - l}{2s} - \frac{w^2 - m}{2w}$$

to make P_8 and P_9 both rational points.

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Indeed we have the following solution,

$$
\begin{aligned}
p &= -(2+11t^2+6t^4)\\
q &= 2(1+t^2)(3+5t^2)\\
s &= 2t(-1+3t^2+t^4)(2+11t^2+6t^4)\\
&\quad \times (8+27t^2+16t^4)\\
w &= 6t(1+t^2)(3+5t^2)^2(2+11t^2+6t^4)\\
u &= (1+t^2)(2+11t^2+6t^4)\\
&\quad \times (16+67t^2+147t^4+115t^6+16t^8)/t.
\end{aligned}
$$

Now let us consider the following elliptic curve,

$$\mathcal{E}_0 \qquad y^2 = x(Ax^2+Bx+C),$$

$\mathcal{E}_0$ is an elliptic curve over $\mathbf{Q}(t)$, and there are 9 $\mathbf{Q}(t)$-rational points $\mathrm{Q}_i(x_i^2, x_i y_i)$ $(1 \le i \le 9)$ and a 2-torsion point $(0,0)$ on $\mathcal{E}_0$.

Then we have the following,

Theorem. $\mathbf{Q}(t)$*-rank of* $\mathcal{E}_0$ *is al least 9.*

Proof. We specialize $t = 2$.

Then we have 9 rational points $\mathrm{R}_1, \cdots, \mathrm{R}_9$ obtained from $\mathrm{Q}_1, \ldots, \mathrm{Q}_9$. By using calculation system PARI, we see that the determinant of the matrix $(\langle \mathrm{R}_i, \mathrm{R}_j \rangle)$ $(1 \le i, j \le 9)$ associated to the canonical height is 13117244956208.81. Since this determinant is non-zero, we see $\mathrm{Q}_1, \ldots, \mathrm{Q}_9$ are independent. □

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