The generalized Whittaker functions for $SU(2,1)$ and the Fourier expansion of automorphic forms

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(Communicated by Heisuke Hironaka, M. J. A., May 12, 2000)

Abstract: Explicit form of Fourier expansion of automorphic forms plays an important role in the theory. Here we investigate the case of $SU(2,1)$ and give an explicit formula of generalized Whittaker functions for the standard representations of the group. Together with a result of [2], we obtain a form of fully developed Fourier expansion of automorphic forms belonging to arbitrary standard representations.

Key words: Generalized Whittaker functions; automorphic forms; Fourier expansion.

Introduction. In the theory of automorphic forms, Fourier expansion of modular forms is a fundamental tool for investigation. In spite of this importance, the theory of Fourier expansion of automorphic forms seems still in very primitive state.

In this short note we give an explicit form of fully developed Fourier expansion of modular forms on $SU(2,1)$. The peculiarity of the case of $SU(2,1)$, different from the case of $SL_2(\mathbf{R})$, is that the maximal unipotent subgroup N is not abelian. It is isomorphic to the Heisenberg group and has infinite dimensional irreducible unitary representations. Together with unitary characters they constitute the unitary dual of N. The Fourier expansion of automorphic forms on $SU(2,1)$ is to consider irreducible decomposition of the restriction $\pi|_N$ of automorphic representations π. Therefore we have to handle those terms which correspond to infinite dimensional representations. As for the terms correspond to unitary characters, Whittaker functions, a quite explicit result is obtained by Koseki-Oda [2]. The remaining problem for our purpose is to consider the generalized Whittaker functions. We investigate these functions for standard representations and obtain an explicit formula for their A-radial parts (Theorem A). Simultaneously, we have the Archimedian local multiplicity one result. Putting all together, we lastly obtain an explicit form of the Fourier expansion of automorphic forms not necessarily holomorphic (Theorem B). This gives the first published form of Fourier expansion of automorphic forms containing the terms corresponding to infinite dimensional representations of N. The full paper is already published in Journal of Mathematical Sciences The University of Tokyo **6**. For details, see [9].

The difficult part of our investigation is the case where π is of the large discrete series representation of $SU(2,1)$, which is most interesting because it is classically unknown.

Notation. We fix notation used through this note. Put $I_{2,1} := \mathrm{diag}(1,1,-1)$. We realize $SU(2,1)$ as $\{g \in SL(3,\mathbf{C}) |\ {}^t\bar{g} I_{2,1} g = I_{2,1}\}$. We denote the group by G and its Lie algebra by $\mathfrak{g}$. And $\mathfrak{g}_{\mathbf{C}}$ is the complexification of $\mathfrak{g}$. Let $G = NAK$ be the Iwasawa decomposition of G. Then in our realization,

$$K := \{\mathrm{diag}(k_1, k_2) \in G \mid k_1 \in U(2),\ k_2 \in U(1),\ k_2 \det k_1 = 1\}.$$

$$A := \left\{ a_r := \begin{pmatrix} \frac{r+r^{-1}}{2} & & \frac{r-r^{-1}}{2} \\ & 1 & \\ \frac{r-r^{-1}}{2} & & \frac{r+r^{-1}}{2} \end{pmatrix} \middle| \ r \in \mathbf{R}_{>0} \right\}.$$

And the maximal unipotent subgroup N is isomorphic to the Heisenberg group $H(\mathbf{R}^2)$ of dimension 3.

We denote the unitary dual of N by $\widehat{N}$ and a standard representation of G by π. In our case π is either a discrete series or a principal series representation.

Let Γ be an arithmetic subgroup of G, containing an element

1991 Mathematics Subject Classification. Primary 11F70; Secondary 11F30, 22E30, 33C15.

$$C^{-1}\begin{pmatrix}1 & & 1\\ & 1 & \\ & & 1\end{pmatrix}C \text{ with } C := \frac{1}{\sqrt{2}}\begin{pmatrix}1 & & 1\\ & \sqrt{2} & \\ 1 & & -1\end{pmatrix}$$

and Φ be an automorphic form on G with respect to Γ belonging to π with K-type (τ, V_τ).

1. Naive Fourier expansion. Let $N_\Gamma := N \cap \Gamma$. As $N_\Gamma\backslash N$ is compact, the decomposition of the right regular representation Reg_N of N is given by

$$L^2(N_\Gamma\backslash N) = \bigoplus_{\sigma\in\widehat{N}} m_\sigma \cdot \mathcal{S}_\sigma,$$

where $(\sigma, \mathcal{S}_\sigma)$ is an N_Γ-invariant unitary representation of N and m_σ is the multiplicity of the representation σ in Reg_N. This reads that a naive Fourier expansion of a V_τ-valued automorphic form Φ along N should be

$$\Phi(ng) = \sum_{\sigma\in\widehat{N}}\sum_{i=1}^{m_\sigma} F^{\pi,\tau}_{\sigma,(i)}(ng),$$

where σ runs through the N_Γ-invariant unitary representations of N. Here $F^{\pi,\tau}_{\sigma,(i)}$ is an $\mathcal{S}^{(i)}_\sigma \otimes_{\mathbf{C}} V_\tau$-valued smooth function in $g \in G$ and $\mathcal{S}^{(i)}_\sigma$ means the i-th copy of $\mathcal{S}_\sigma$ $(i = 1, \ldots, m_\sigma)$.

Famous Stone-von Neumann theorem tells that $\widehat{N}$ is exhausted by unitary characters $\psi_{u,v}$ parameterized by $(u, v) \in \mathbf{R}^2$ and infinite dimensional irreducible unitary representations ρ_{ψ_s} determined by their nontrivial central characters ψ_s's. When σ is a unitary character, its multiplicity $m_{\psi_{u,v}}$ is one. Hence

$$\Phi(ng) = \sum_{\psi_{u,v}} F^{\pi,\tau}_{\psi_{u,v}}(ng) + \sum_{\rho_{\psi_s}}\sum_{i=1}^{m_\rho} F^{\pi,\tau}_{\rho_{\psi_s},(i)}(ng).$$

The functions $F^{\pi,\tau}_{\psi_{u,v}}$ are the Whittaker functions and we study $F^{\pi,\tau}_{\rho_{\psi_s},(i)}$.

2. Generalized Whittaker functions. Naive formulation of the problem is to investigate intertwiners in $\mathrm{Hom}_N(\pi|_N, \rho_\psi)$ which is isomorphic to $\mathrm{Hom}_G(\pi, \mathrm{Ind}_N^G\,\rho_\psi)$ by Frobenius reciprocity. But this fails in general, because the intertwining space in question is infinite dimensional. And so we introduce a larger group R containing N to formulate the problem rightly.

Let P be a minimal parabolic subgroup of G with Levi decomposition: $P = L \ltimes N$, where L is the reductive part of P (*i.e.* Levi subgroup). The action of L on N by conjugation induces its action on $\widehat{N}$. Let S be the stabilizer of ρ_ψ in L. Because the equivalence class of ρ_ψ in $\widehat{N}$ is completely determined by its central character, S is the centralizer of $Z(N)$ and independent of ρ_ψ. It is easily checked that S is isomorphic to $SO(2)$. Put $R := S \ltimes N$.

We extend representation ρ_ψ of N to R by using the Weil representation $\omega_\psi\times\rho_\psi\colon\ \widetilde{Sp}_1(\mathbf{R})\ltimes H(\mathbf{R}^2) \to \mathrm{Aut}(L^2(\mathbf{R}))$. The semidirect product $R = S \ltimes N$ is regarded as a subgroup of $\widetilde{Sp}_1(\mathbf{R}) \ltimes H(\mathbf{R}^2)$. Let $\widetilde{R}$ be the pullback $\widetilde{S} \ltimes N \cong \widetilde{SO}(2) \ltimes H(\mathbf{R}^2)$ of R by the covering $pr \times id : \widetilde{Sp}_1(\mathbf{R}) \ltimes H(\mathbf{R}^2) \twoheadrightarrow SL_2(\mathbf{R}) \ltimes H(\mathbf{R}^2)$. Then tensoring an odd character $\widetilde{\chi}_\mu$ of $\widetilde{SO}(2)$ to $(\omega_\psi \times \rho_\psi)|_{\widetilde{R}}$, we have a representation $\widetilde{\chi}_\mu \otimes (\omega_\psi \times \rho_\psi)|_{\widetilde{R}}$ of $\widetilde{R}$. We denote this representation by $(\eta, L^2(\mathbf{R}))$. A character $\widetilde{\chi}_\mu$ of $\widetilde{SO}(2)$ is called odd, if it does not factor through the covering $\widetilde{SO}(2) \twoheadrightarrow SO(2)$. That is, the parameter is of the form $\mu = m + 1/2$ $(m \in \mathbf{Z})$.

Here is a diagram explaining the above construction

$$\begin{array}{ccc}
\widetilde{R} \hookrightarrow & \widetilde{Sp}_1(\mathbf{R}) \ltimes H(\mathbf{R}^2) & \xrightarrow{\omega_\psi\times\rho_\psi} \mathrm{Aut}(L^2(\mathbf{R}))\\
\downarrow & \downarrow{\scriptstyle pr\times id} & \\
R \hookrightarrow & SL_2(\mathbf{R}) \ltimes H(\mathbf{R}^2). &
\end{array}$$

Definition. We call $\mathrm{Hom}_{(\mathfrak{g}_{\mathbf{C}},K)}(\pi, \mathrm{Ind}_R^G\,\eta) =: I_{\pi,\eta}$ *the space of the algebraic generalized Whittaker functionals.* Here we identify the underlining $(\mathfrak{g}_{\mathbf{C}}, K)$-module of π with π itself.

In order to investigate algebraic generalized Whittaker functionals $l \in I_{\pi,\eta}$, we specify a K-type of π to (τ, V_τ). Choose an injection $\iota_\tau : \tau^* \hookrightarrow \pi|_K$, and pullback l by ι_τ. We study functions $F \in C^\infty_{\eta,\tau}(R\backslash G/K) := \{\varphi : G \to \mathcal{S}(\mathbf{R}) \otimes_{\mathbf{C}} V_\tau \mid \varphi \text{ is } C^\infty, \varphi(rgk) = \eta(r)\tau(k)^{-1}.\varphi(g)\}$ representing $\iota_\tau^* l \in \mathrm{Hom}_K(\tau^*, \mathrm{Ind}_R^G\,\eta|_K) =: I^\tau_{\pi,\eta}$. By definition, $\iota^*_\tau l(v^*)(g) = \langle v^*, F(g)\rangle_K$, $v^* \in V_{\tau^*}$. Here $\langle\ ,\ \rangle_K$ means the canonical pairing of K-modules V_{τ^*} and V_τ, τ^* the contragredient to τ. We call the above function F *the algebraic generalized Whittaker function associated to π with K-type τ.* This modification corresponds to a fine expansion $F_\rho = \sum_\chi F_\eta$.

Let (μ_1, μ_2) be the highest weight of τ then we can realize V_τ as $\mathbf{C}$-span of $\{v^\mu_k\}_{k=0}^{d_\mu}$ $(d_\mu := \mu_1 - \mu_2)$ with $\mathfrak{k}_{\mathbf{C}}$ action $\tau_\mu(Z)v^\mu_k = (\mu_1 + \mu_2)v^\mu_k$, $\tau_\mu(H'_{12})v^\mu_k = (2k - d_\mu)v^\mu_k$, $\tau_\mu(H'_{13})v^\mu_k = (k + \mu_2)v^\mu_k$, $\tau_\mu(X_{\beta_{12}})v^\mu_k = (k + 1)v^\mu_{k+1}$, $\tau_\mu(X_{\beta_{21}})v^\mu_k = (k - d_\mu - 1)v^\mu_{k-1}$. Here $H'_{12} = \mathrm{diag}(1, -1, 0)$, $H'_{13} = \mathrm{diag}(1, 0, -1)$, $Z = 2H'_{13} - H'_{12}$. As for η we take $\{h_j \mid j = 0, 1, 2, \ldots\}$ as its basis. Here h_j is the j-

th Hermite function. Note F is determined by the A-radial part $F|_A \in C^\infty(A;\mathcal{S}(\mathbf{R})\otimes_{\mathbf{C}} V_\tau) := \{\phi : A \to \mathcal{S}(\mathbf{R})\otimes_{\mathbf{C}} V_\tau \mid C^\infty\text{-function}\}$ because of its equivariance and the Iwasawa decomposition. Expand $F|_A$ with respect to basis of $\mathcal{S}(\mathbf{R})$ and V_τ: $F|_A(a) = \sum_{j=0}^{\infty}\sum_{k=0}^{d_\mu} c_{jk}(a)\big(h_j\otimes v_k^\mu\big)$. The compatibility of S-action and K-action implies the vanishing of many coefficients c_{jk}.

Lemma. *The restriction of F in $C^\infty(A;\mathcal{S}(\mathbf{R})\otimes_{\mathbf{C}} V_\tau)$ is zero unless $(-\lambda_1+2\lambda_2)/3\in\mathbf{Z}$ and $(-\lambda_1+2\lambda_2)3\ge(1/2)+\mu$. When the condition above holds, the A-radial part is written as*

$$F|_A(a_r) = \sum_{k=0}^{d_\lambda} c_k(a_r)\big(h_{j_k}\otimes v_k^\lambda\big),$$

where c_k's are C^∞-functions on A and the index j_k is given by $j_k = -k + (2\lambda_1-\lambda_2)/3-(1/2)-\mu$.

Proof. Calculate $F|_A(mam^{-1})$, $m\in S = M$, $a\in A$ in two ways. First, since $M = Z_K(A)$, $mam^{-1} = a$, therefore $F|_A(mam^{-1}) = F|_A(a)$. Second, because $M = S\subset R$ and $F|_A$ is a function which comes from $F\in C^\infty_{\eta,\tau_\lambda}(R\backslash G/K)$, $F|_A(mam^{-1}) = \eta(m)\tau_\lambda(m)F|_A(a)$. Compare coefficients of these two, we have the assertion. □

3. The case of discrete series representations. Let $\mathfrak{t}$ be the compact Cartan subalgebra $\{\mathrm{diag}(\sqrt{-1}h_1,\sqrt{-1}h_2,\sqrt{-1}h_3)\mid h_i\in\mathbf{R},\ h_1+h_2+h_3=0\}$ and β_{ij} be a linear form on $\mathfrak{t}_{\mathbf{C}}$ defined by $\beta_{ij}:\mathrm{diag}(t_1,t_2,t_3)\mapsto t_i-t_j$. Then Harish-Chandra parameterization claims that there is a bijection between the set Ξ of all Σ-regular Σ_c^+-dominant T-integral weights and the set $\widehat{G}_d$ of all equivalence classes of discrete series representations of G. Here Σ and Σ_c^+ denote the root system associated to $(\mathfrak{g}_{\mathbf{C}},\mathfrak{t}_{\mathbf{C}})$ given by $\{\beta_{ij}\mid i\ne j,\ 1\le i,j\le 3\}$ and the positive compact root system fixed as $\{\beta_{12}\}$ respectively. We can identify Ξ with $\{\Lambda=(\Lambda_1,\Lambda_2)\in\mathbf{Z}^{\oplus 2}\mid \Lambda_1>\Lambda_2,\ \Lambda_1\Lambda_2\ne 0\}$. We fix Σ_c^+ compatible positive systems as $\Sigma_I^+=\{\beta_{12},\beta_{13},\beta_{23}\}$, $\Sigma_{II}^+=\{\beta_{12},\beta_{32},\beta_{13}\}$, $\Sigma_{III}^+=\{\beta_{12},\beta_{32},\beta_{31}\}$. Then Ξ decomposes into three disjoint subsets $\Xi_I^+=\{\Lambda\in\Xi\mid\Lambda_1>\Lambda_2>0\}$, $\Xi_{II}^+=\{\Lambda\in\Xi\mid\Lambda_1>0>\Lambda_2\}$, $\Xi_{III}^+=\{\Lambda\in\Xi\mid 0>\Lambda_1>\Lambda_2\}$ correspond to Σ_J^+. We call representations parameterized by Ξ_I^+ (resp. Ξ_{III}^+) the holomorphic (resp. the anti-holomorphic) discrete series representations. In the remaining case, representations whose Harish-Chandra parameters Λ's belong to Ξ_{II}^+ are the large discrete series representations in the sense of Vogan [7].

The Blattner formula tells us the K-type decomposition of the discrete series π_Λ as follows. $\pi_\Lambda|_K=\oplus_{\mu\in L_T^+(\Lambda)}[\pi_\Lambda:\tau_\mu]\tau_\mu$, where the set $L_T^+(\Lambda)$ of parameters of the K-types of is given by $\{\lambda+m_1\beta_{13}+m_2\beta_{23}\}$ when $\Lambda\in\Xi_I^+$, $\{\lambda+m_1\beta_{13}+m_2\beta_{32}\}$ when $\Lambda\in\Xi_{II}^+$, $\{\lambda+m_1\beta_{31}+m_2\beta_{32}\}$ when $\Lambda\in\Xi_{III}^+$. Here m_1, m_2 run through $\mathbf{Z}_{\ge 0}$. And λ is the highest weight of *the minimal K-type* of π_Λ and called the Blattner parameter. About the multiplicity we remark that all $[\pi_\Lambda:\tau_\lambda]$ is one.

Define $\nabla_\tau : \mathrm{Ind}_K^G\tau \to \mathrm{Ind}_K^G(\tau\otimes\mathrm{Ad}_{\mathfrak{p}_{\mathbf{C}}})$ as $\varphi\mapsto\sum_{i=1}^4 R_{X_i}\varphi\otimes X_i$, where $\{X_i\ (i=1,\dots,4)\}$ is an orthonormal basis of $\mathfrak{p}$ with respect to the Killing form and R_X means the right defferential by X. Clebsch-Gordan's theorem tells the following decomposition $\tau_\mu\otimes\mathrm{Ad}_{\mathfrak{p}_{\mathbf{C}}}\cong\oplus_{\beta\in\Sigma_n}\tau_{\mu-\beta}$. Denote the projector onto $\tau_{\mu-\beta}$ by $p_{-\beta}$. Here is a variant of a result of Yamashita which is fundamental for our purpose.

Proposition 1 ([8])**.** *Assume Λ is far from walls, then the image of $\mathrm{Hom}_{(\mathfrak{g}_{\mathbf{C}},K)}(\tau_\lambda^*,\mathrm{Ind}_R^G\eta)$ in $C^\infty_{\eta,\tau_\lambda}(R\backslash G/K)$ is characterized by*

$$\text{(D):}\qquad (p_{-\beta}\cdot\nabla_{\tau_\lambda}).F=0\quad(\forall\beta\in\Sigma_J^+\cap\Sigma_n).$$

Naturally our generalized Whittaker functions satisfies the above system of defferential equations (D).

Restricting to the A-radial part, we write down the action of $p_{-\beta}\cdot\nabla_\tau$ by the coefficient functions. Denote the Euler operator $r(d/dr)$ by ∂.

Proposition 2. *Expand $2(p_{-\beta}\cdot\nabla_\tau)F|_A(a_r)$ as $\sum_{k=0}^{d_{\lambda-\beta}}c_k^{-\beta}(a_r)\big(h_{j_k}\otimes v_k^{\lambda-\beta}\big)$ then the coefficient functions are given as*

$$\begin{aligned}
c_k^{-\beta_{23}} &= (d_\lambda-k+1)(\partial+k-\lambda_2-2r^2s)c_k\\
&\quad +k(1+2s)rc_{k-1}/\sqrt{2},\\
c_k^{-\beta_{13}} &= (\partial+k-2d_\lambda-\lambda_2-1-2r^2s)c_{k+1}\\
&\quad -(1+2s)rc_k/\sqrt{2},\\
c_k^{-\beta_{32}} &= (\partial-k+\lambda_2-2+2r^2s)c_k\\
&\quad -\sqrt{2}(1+2s)(j_k+1)rc_{k+1},\\
c_k^{-\beta_{31}} &= k(\partial-k+\lambda_2+2d_\lambda+1+2r^2s)c_{k-1}\\
&\quad +(d_\lambda-k+1)\sqrt{2}(1+2s)(j_k+1)rc_k.
\end{aligned}$$

We omitted the variable a_r of c_k's.

Now let us define *the generalized Whittaker model for the representation π of G with K-type τ* as $Wh_\eta^\tau(\pi):=\{F\in C^\infty_{\eta,\tau}(R\backslash G/K)\mid F|_A(a_r)$ is of moderate growth when $r\to\infty$, $l(v^*)=\langle v^*,F(\cdot)\rangle_K$,

$l \in I_{\pi,\eta}$, $v^* \in V_\tau^*\}$. We call the elements in the space above *the generalized Whittaker functions associated to the representation* π *with* K*-type* τ.

Let $\mathcal{A}_\eta(R\backslash G)$ be a $(\mathfrak{g}_{\mathbf{C}}, K)$-submodule of $\mathrm{Ind}_R^G \eta$ defined by $\{f \in \mathrm{Ind}_R^G \eta \mid c_{f,h}$ is right K-finite and $c_{f,h}|_A(a_r)$ is of moderate growth when $r \to \infty$, $\forall h \in (\eta, \mathcal{S}(\mathbf{R}))\}$, where $c_{f,h}(g) := (f(g), h)_\eta$.

Theorem A. *The dimension of the space* $\mathrm{Hom}_{(\mathfrak{g}_{\mathbf{C}},K)}(\pi_\Lambda, \mathcal{A}_\eta(R\backslash G))$ *of generalized Whittaker functionals is at most one. Moreover*

$$\dim_{\mathbf{C}} \mathrm{Hom}_{(\mathfrak{g}_{\mathbf{C}},K)}(\pi_\Lambda, \mathcal{A}_\eta(R\backslash G)) = 1$$

if and only if $\ell_\lambda = (2\lambda_1 - \lambda_2)/3 \in \mathbf{Z}$, $\ell_\lambda + (1/2) + \mu \le 0$. *Under this condition, the minimal* K*-type generalized Whittaker model* $Wh_\eta^{\tau_\lambda}(\pi_\Lambda)$ *has a base* $F_\eta^{\tau_\lambda}$ *whose* A*-radial part is given as follows.*

i-1) π_Λ : *a large discrete series*

$$F_\eta^{\tau_\lambda}(a_r) = \sum_{k=0}^{d_\lambda} r^{d_\lambda+1} W_{\kappa,(k-\lambda_1)/2}(2|s|r^2)\Big(h_{j_k} \otimes v_k^\lambda\Big),$$

where $\kappa = \{-(\lambda_1 - k - 1)s - (j_k+1)(2s+1)^2/4\}/2|s|$, $s \in \mathbf{R}\backslash\{0\}$.

i-2) π_Λ : *a holomorphic discrete series*

$$F_\eta^{\tau_\lambda}(a_r) = \sum_{k=0}^{d_\lambda} r^{\lambda_2+k} e^{sr^2} \cdot \Big(h_{j_k} \otimes v_k^\lambda\Big),$$

where $s < 0$.

i-3) π_Λ : *an antiholomorphic discrete series*

$$F_\eta^{\tau_\lambda}(a_r) = \sum_{k=0}^{d_\lambda} r^{-\lambda_2-k} e^{-sr^2} \cdot \Big(h_{j_k} \otimes v_k^\lambda\Big),$$

where $s > 0$.

Proof. In the large discrete series case $\Lambda \in \Xi_{II}$, the system (D) of defferential equations, which characterizes the minimal K-type F, is equivalent to $c_k^{-\beta_{13}} = 0$, $c_k^{-\beta_{32}} = 0$. Eliminating difference term, we have a defferential equation of second order $\{\partial^2 - (2d_\lambda + 4)\partial + G_k(r)\}.c_k(a_r) = 0$ with $G_k(r) = -4s^2r^4 - \{4(\lambda_2 - k + d_\lambda - 1)s + (j_k+1)(1+2s)^2\}r^2 - (k - 2d_\lambda - \lambda_2 - 2)(k - \lambda_2 + 2).$, which turns into the classical Whittaker defferential equation. Under the growth condition of $Wh_\eta^{\tau_\lambda}$, we can find unique solution $c_k(a_r) = (\text{const.}) \times r^{d_\lambda+1} W_{\kappa,(k-\lambda_1)/2}(2|s|r^2)$, $k = 0, \ldots, d_\lambda$. In other cases $\Lambda \in \Xi_I$, Ξ_{III}, we obtain a first order defferential equation for c_k whose solution is essentially given by exponential function. $\square$

4. The case of principal series representations. Let $P = NAM$ be the Langlands decomposition of P. For characters $e^\nu : a_r \mapsto r^{\nu+2}$ ($\nu \in \mathbf{C}$) of A and $\chi_{\lambda_0} : \mathrm{diag}(e^{i\theta}, e^{-2i\theta}, e^{i\theta}) \mapsto e^{i\lambda_0\theta}$ ($\lambda_0 \in \mathbf{Z}$) of M, the induced representation $\pi_{\lambda_0,\nu} = \mathrm{Ind}_P^G(1_N \otimes e^\nu \otimes \chi_{\lambda_0})$ is called the principal series representation of G. By the Frobenius duality, the K-type decomposition is given by $\pi_{\lambda_0,\nu}|_K = \oplus_{\mu \in L_T^+(\lambda_0)} \tau_\mu$ with $L_T^+(\lambda_0) = \{(-\lambda_0, -\lambda_0) + m_1\beta_{13} + m_2\beta_{32} \mid m_1, m_2 \ge 0 \in \mathbf{Z}\}$. The K-type $\tau_0 := \tau_{(-\lambda_0,-\lambda_0)}$ only has dimension one and called *the corner* K*-type of* $\pi_{\lambda_0,\nu}$. In this case, thanks to the one dimensionality of τ_0, we can obtain the defferential equation satisfied by the generalized Whittaker function only by calculating the Casimir operator.

Proposition 3. *Let* F *be the generalized Whittaker function corresponds to* $l \in I^{\tau_0}_{\pi_{\lambda_0,\nu},\eta}$. *Then the coefficient function* c_0 *of* $F|_A$ *satisfies the defferential equation* $\{\partial^2 - 4\partial + G(r)\}.c_0(a_r) = 0$ *with* $G(r) = -4s^2r^4 + \{-4\lambda_0 s - (2j_0+1)(1+4s^2)\}r^2 - (\nu^2 - 4)$.

Theorem A. *The irreducible principal series representation* $\pi_{\lambda_0,\nu}$ *has multiplicity one property* i.e.

$$\dim_{\mathbf{C}} \mathrm{Hom}_{(\mathfrak{g}_{\mathbf{C}},K)}(\pi_{\lambda_0,\nu}, \mathcal{A}_\eta(R\backslash G)) = 1$$

if and only if $(\lambda_0/3) - \mu - (1/2) \in \mathbf{Z}_{\ge 0}$. *Under this condition, the corner* K*-type generalized Whittaker model* $Wh_\eta^{\tau_0}(\pi_{\lambda_0,\nu})$ *has a base* $F_\eta^{\tau_0}$ *whose* A*-radial part is given by*

$$F_\eta^{\tau_0}(a_r) = rW_{\kappa,\nu/2}(2|s|r^2) \cdot \Big(h_{j_0} \otimes v_0\Big),$$

where $\kappa = \{-\lambda_0 s - (2j_0+1)(4s^2+1)/4\}/2|s|$ *and the index* j_0 *is given by* $j_0 = (\lambda_0/3) - \mu - (1/2)$.

5. The Fourier expansion. Now we can give an explicit form of the Fourier expansion of an automorphic form belonging to an arbitrary standard representation π with special K-type.

Theorem B. *Let* Φ *be as above.*

i) *When* π *is a discrete series representation* π_Λ *take the minimal* K*-type* τ_λ *as* τ.

i-1) *The case of large discrete series*

$$\Phi(na_r) = C^\Phi_{0,0} \cdot r^{d_\lambda+2} \cdot 1_N(n) v^\lambda_{\lambda_1} + \sum_{(\ell,\ell') \in \mathbf{Z}^2}{}' \Big(\sum_{k=0}^{d_\lambda} C^{\Phi,k}_{\ell,\ell'} \cdot r^{d_\lambda+3/2} W_{0,k-\lambda_1}(4\pi L r) \times \psi_{2\pi\ell,2\pi\ell'}(n) v_k^\lambda\Big)$$

$$+\sum_{\ell\in\mathbf{Z}}{}' \sum_{i=1}^{2|\ell|}\Big(\sum_{k=0}^{d_\lambda}\sum_{\mu\in(1/2)\mathbf{Z}\setminus\mathbf{Z}} C^{\Phi,\,k}_{\mu,\ell,\,(i)} \times r^{d_\lambda+1}W_{\kappa,(k-\lambda_1)/2}(4\pi|\ell|r^2)\cdot\theta^{\ell,\,(i)}_{j_k}(n)v^\lambda_k\Big),$$

where $\kappa=\{-(\lambda_2-k+d_\lambda-1)2\pi\ell-(j_k+1)(4\pi\ell+1)^2/4\}/4\pi|\ell|$ *and* $L=\sqrt{\ell^2+(\ell')^2}$.

i-2) *The case of holomorphic discrete series*

$$\Phi(na_r)=\sum_{-\ell=0}^{\infty}\sum_{i=1}^{2|\ell|}\Big(\sum_{k=0}^{d_\lambda}\sum_{\mu\in(1/2)\mathbf{Z}\setminus\mathbf{Z}} C^{\Phi,\,k}_{\mu,\ell,\,(i)} \times r^{\lambda_2+k}e^{2\pi\ell r^2}\cdot\theta^{\ell,\,(i)}_{j_k}(n)v^\lambda_k\Big).$$

i-3) *The case of antiholomorphic discrete series*

$$\Phi(na_r)=\sum_{\ell=0}^{\infty}\sum_{i=1}^{2\ell}\Big(\sum_{k=0}^{d_\lambda}\sum_{\mu\in(1/2)\mathbf{Z}\setminus\mathbf{Z}} C^{\Phi,\,k}_{\mu,\ell,\,(i)} \times r^{-\lambda_2-k}e^{-2\pi\ell r^2}\cdot\theta^{\ell,\,(i)}_{j_k}(n)v^\lambda_k\Big).$$

In this case, the index μ *runs over half integers which satisfy* $j_k\geq 0$.

ii) *When* π *is a principal series representation* $\pi_{\lambda_0,\nu}$, *take the corner* K*-type* τ_0 *as* τ.

$$\begin{aligned}\Phi(na_r)&=C^{\Phi}_{0,0}\cdot r^{\nu+2}\cdot 1_N(n)v_0\\&+\sum_{(\ell,\ell')\in\mathbf{Z}^2}{}' C^{\Phi}_{\ell,\ell'}\cdot r^{3/2}W_{0,\nu}(4\pi Lr)\cdot\psi_{2\pi\boldsymbol{\ell}}(n)v_0\\&+\sum_{\ell\in\mathbf{Z}}{}'\sum_{i=1}^{2|\ell|}\Big(\sum_{\mu\in(1/2)\mathbf{Z}\setminus\mathbf{Z}} C^{\Phi}_{\mu,\ell,\,(i)}\cdot rW_{\kappa,(\nu/2)}(4\pi|\ell|r^2)\times\theta^{\ell,\,(i)}_{j_0}(n)v_0\Big),\end{aligned}$$

where $\kappa=\{-\lambda_0 2\pi\ell-(2j_0+1)(16\pi^2\ell^2+1)^2/4\}/4\pi|\ell|$. *In this case, the index* μ *runs over half integers such that* $\mu\leq(\lambda_0/3)-(1/2)$.

Here the generalized theta functions $\theta^{\ell,\,(i)}_j$ *are given by*

$$\theta^{\ell,\,(i)}_j\Big(\exp(xE_{2,+}+yE_{2,-}+zE_1)\Big)=\sum_{k\in\mathbf{Z}}h_j(x+d_k)\cdot\mathrm{e}[(i+2|\ell|k)y+\ell xy+\ell z],$$

with $d_k=(i+2|\ell|k)/(2\ell)$ *and* $\mathrm{e}[X]=e^{2\pi\sqrt{-1}X}$. *We call* $C^{\Phi}_{\ell,\ell'}$*'s,* $C^{\Phi,\,k}_{\mu,\ell,\,(i)}$*'s the Fourier coefficients of* Φ.

Proof. Almost all is clear from the result of [2] and Theorem A. The remaining task for us is construction of the base $\{\theta^{\rho_{\psi_s}\,(i)}_j\}_{j\in\mathbf{N}}$ of the i-th copy $\mathcal{S}^{(i)}_{\rho_{\psi_s}}\subset L^2(N_\Gamma\backslash N)$ of $\mathcal{S}_{\rho_{\psi_s}}$ corresponding to $\{h_j\}$ by an N-intertwiner $T:\mathcal{S}(\mathbf{R})\to L^2(N_\Gamma\backslash N)$. For the purpose, we only have to write down the Hermite descending operator by elements of $U(\mathfrak{n})$ and translate it by T. Then we have the defferential equation which is satisfied by the image $\theta^{\rho_{\psi_s}\,(i)}_0=T(h_0)$ of h_0. Using quasi-periodicity of $\theta^{\rho_{\psi_s}\,(i)}_0$ come from N_Γ-invariantness, we can solve the defferential equation and obtain the explicit form of $\theta^{\rho_{\psi_s}\,(i)}_0$. By the assumption on Γ, we have the multiplicity $m_\rho=2|\ell|$. Next, by using the raising operator recursively, we obtain the form of $\theta^{\rho_{\psi_s}\,(i)}_j=T(h_j)$ as above. □

Remark. The case of holomorphic or antiholomorphic discrete series, our discussion of the Fourier expansion of automorphic form accords with the Fourier-Jacobi expansion obtained by Shintani [6], Murase-Sugano [3].

Acknowledgement. The author would like to express my sincere gratitude to Prof. Takayuki Oda for his constant guidance and warm encouragement.

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