"Hasse principle" for $GL_n(D)$

By Hideo WADA

Department of Mathematics, Sophia University, 7-1, Kioicho, Chiyoda-ku, Tokyo 102-8554

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Abstract: Let D be a Euclidian domain, G be a $GL_n(D)$ and f be an endomorphism of G which preserves conjugacy classes. Then we shall prove f must be an inner automorphism, namely G enjoys the Hasse principle.

Key words: Hasse principle; general linear group.

1. Notation and result. Let D be a Euclidean domain, n be a positive integer and G be the $GL_n(D)$. Put $\varepsilon = (-1)^{n-1}$ and we define in G

$$S = \begin{pmatrix} 0 & \cdots & \cdots & 0 & \varepsilon \\ 1 & \ddots & & & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix},$$

$$T_\nu = \begin{pmatrix} 1 & \nu & 0 & \cdots & 0 \\ 0 & 1 & 0 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}, \quad T = T_1,$$

$$D_\mu = \begin{pmatrix} 1+\mu & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}, \quad \begin{matrix} 1+\mu = \text{unit}, \\ \mu \neq 0. \end{matrix}$$

Then G is generated by S, T_ν and D_μ (cf.[1]). Using this fact T. Ono([2]) proved that $GL_2(D)$ enjoys the Hasse principle. In this paper we shall prove more generally the following theorem:

Theorem. *For any positive integer n, $GL_n(D)$ enjoys the Hasse principle.*

2. Proof of the theorem. Let $f(x)$ be an endomorphism which satisfies $f(x) \sim x$ (conjugate in G) for each $x \in G$. We shall prove that f must be an inner automorphism.

In [3], we proved that $SL_n(D)$ enjoys the Hasse principle. So we may assume $f(S) = S$, $f(T_\nu) = T_\nu$ (Of course we must change that proof a little). Let $f(D_\mu) = M^{-1}D_\mu M$ where

$$M = \begin{pmatrix} a_1 & \cdots & a_n \\ & & \\ & * & \end{pmatrix}, \quad M^{-1} = \begin{pmatrix} x_1 & & \\ \vdots & * & \\ x_n & & \end{pmatrix}.$$

Then

$$\begin{aligned} f(D_\mu) &= M^{-1}(E + \mu E_{11})M \\ &= E + \mu \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} (a_1 \cdots a_n) \end{aligned}$$

where E_{ij} is the matrix unit whose ij-element is 1 and the other elements are 0. Put $y_i = \mu x_i$ $(1 \leq i \leq n)$. As

$$SD_\mu \sim f(SD_\mu) = f(S)f(D_\mu) = SM^{-1}D_\mu M$$

their charactaristic polynomials are equal.

$$\begin{aligned} &|xE - SM^{-1}D_\mu M| \\ &= \left| xE - S - \begin{pmatrix} \varepsilon y_n \\ y_1 \\ \vdots \\ y_{n-1} \end{pmatrix} (a_1 \ \cdots \ a_n) \right| \\ &= \begin{vmatrix} x & 0 & \cdots & 0 \\ & & * & \end{vmatrix} + \begin{vmatrix} 0 & 0 & \cdots & -\varepsilon \\ & & * & \end{vmatrix} \\ &\quad + \begin{vmatrix} -\varepsilon y_n a_1 & \cdots & -\varepsilon y_n a_n \\ & * & \end{vmatrix}. \end{aligned}$$

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Put these terms a, b and c. Then

$$c = -\varepsilon y_n \times \begin{vmatrix} a_1 & a_2 & a_3 & \cdots \\ -1 - y_1 a_1 & x - y_1 a_2 & -y_1 a_3 & \cdots \\ -y_2 a_1 & -1 - y_2 a_2 & x - y_2 a_3 & \cdots \\ \vdots & \vdots & \ddots & \ddots \end{vmatrix}$$

$$= -\varepsilon y_n \begin{vmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ -1 & x & 0 & \cdots & 0 \\ 0 & -1 & x & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & x \end{vmatrix}$$

$$= -\varepsilon y_n (a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n).$$

$$a = x|A_{n-1}|$$

$$= x \begin{vmatrix} x - y_1 a_2 & -y_1 a_3 & -y_1 a_4 & \cdots \\ -1 - y_2 a_2 & x - y_2 a_3 & -y_2 a_4 & \cdots \\ -y_3 a_2 & -1 - y_3 a_3 & x - y_3 a_4 & \cdots \\ \vdots & \vdots & \ddots & \ddots \end{vmatrix}$$

$$= x^2 |A_{n-2}| - y_1 x \times \begin{vmatrix} a_2 & a_3 & a_4 & \cdots \\ -1 - y_2 a_2 & x - y_2 a_3 & -y_2 a_4 & \cdots \\ -y_3 a_2 & -1 - y_3 a_3 & x - y_3 a_4 & \cdots \\ \vdots & \vdots & \ddots & \ddots \end{vmatrix}$$

$$= x^2 |A_{n-2}| - y_1 x \begin{vmatrix} a_2 & a_3 & \cdots & a_n \\ -1 & x & & 0 \\ & \ddots & \ddots & \\ 0 & & -1 & x \end{vmatrix}$$

$$= x^2 |A_{n-2}| - y_1 x (a_2 x^{n-2} + \cdots + a_n).$$

Using induction we have

$$a = x^n - y_1 (a_2 x^{n-1} + \cdots + a_n x) - y_2 (a_3 x^{n-1} + \cdots + a_n x^2) - \cdots - y_{n-1} a_n x^{n-1}.$$

$$b = (-1)^{n+1} (-\varepsilon) \times \begin{vmatrix} -1 - y_1 a_1 & x - y_1 a_2 & -y_1 a_3 & \cdots \\ -y_2 a_1 & -1 - y_2 a_2 & x - y_2 a_3 & \cdots \\ -y_3 a_1 & -y_3 a_2 & -1 - y_3 a_3 & \ddots \\ \vdots & \vdots & \vdots & \ddots \end{vmatrix}$$

$$= -|B_{n-1}|.$$

$$|B_{n-1}| = \begin{vmatrix} -1 & \\ 0 & \\ \vdots & * \\ 0 & \end{vmatrix} + \begin{vmatrix} -y_1 a_1 & \\ -y_2 a_1 & \\ \vdots & * \\ -y_{n-1} a_1 & \end{vmatrix}$$

$$= -|B_{n-2}| + |C_{n-1}|.$$

$$|C_{n-1}| = -a_1 \begin{vmatrix} y_1 & x & 0 & \cdots & 0 \\ y_2 & -1 & x & \ddots & \vdots \\ y_3 & 0 & -1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & x \\ y_{n-1} & 0 & \cdots & 0 & -1 \end{vmatrix}$$

$$= (-1)^{n-1} a_1 (y_1 + y_2 x + \cdots + y_{n-1} x^{n-2}).$$

Using induction we have

$$b = -\varepsilon a_1 (y_1 + y_2 x + \cdots + y_{n-1} x^{n-2}) - \varepsilon a_2 (y_2 + y_3 x + \cdots + y_{n-1} x^{n-3}) - \cdots - \varepsilon a_{n-1} y_{n-1} - \varepsilon.$$

Put all together we have

$$|xE - SM^{-1} D_\mu M|$$
$$= x^n - \sum_{i=1}^{n} (y_1 a_{i+1} + y_2 a_{i+2} + \cdots + y_{n-i} a_n + \varepsilon y_{n-i+1} a_1 + \cdots + \varepsilon y_n a_i) x^{n-i} - \varepsilon.$$

If we put $M = E$, then we have

$$|xE - SD_\mu| = x^n - (\mu + 1)\varepsilon.$$

Comparing the coefficients of two polynomials, we have

(1)
$$\begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ \varepsilon x_2 & \varepsilon x_3 & \cdots & \varepsilon x_n & x_1 \\ \varepsilon x_3 & \cdots & \varepsilon x_n & x_1 & x_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \varepsilon x_n & x_1 & x_2 & \cdots & x_{n-1} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Next we use

$$TSD_\mu \sim f(TSD_\mu) = TSM^{-1} D_\mu M.$$

As $T = E + E_{12}$, we have

$$|xE - TSM^{-1} D_\mu M| = |xE - SM^{-1} D_\mu M| + |F_n|.$$

$$|F_n| = \begin{vmatrix} -1 - y_1 a_1 & -y_1 a_2 & \cdots \\ -1 - y_1 a_1 & x - y_1 a_2 & \cdots \\ \vdots & \ddots & \ddots \end{vmatrix}$$

$$= \begin{vmatrix} -1-y_1a_1 & -y_1a_2 & -y_1a_3 & \cdots \\ 0 & x & 0 & \cdots \\ -y_2a_1 & -1-y_2a_2 & x-y_2a_3 & \cdots \\ \vdots & \vdots & \ddots & \ddots \end{vmatrix}$$

$$= x \begin{vmatrix} -1-y_1a_1 & -y_1a_3 & -y_1a_4 & \cdots \\ -y_2a_1 & x-y_2a_3 & -y_2a_4 & \cdots \\ -y_3a_1 & -1-y_3a_3 & x-y_3a_4 & \cdots \\ \vdots & \vdots & \ddots & \ddots \end{vmatrix}$$

$$= x \begin{vmatrix} -y_1a_1 & -y_1a_3 & \cdots & -y_1a_n \\ & * & & \end{vmatrix}$$

$$+x \begin{vmatrix} -1 & 0 & \cdots & 0 \\ & * & & \end{vmatrix}$$

$$= -y_1x \begin{vmatrix} a_1 & a_3 & a_4 & \cdots \\ 0 & x & 0 & \cdots \\ 0 & -1 & x & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{vmatrix} - x|A_{n-2}|$$

$$= -y_1a_1x^{n-1} - x^{n-1} + y_2(a_3x^{n-2} + \cdots + a_nx) + \cdots + y_{n-1}a_nx^{n-2}.$$

If we put $M = E$, then we have

$$|xE - TSD_\mu| = x^n - (\mu+1)\varepsilon - (\mu+1)x^{n-1}.$$

Therefore we have

(2)

$$\begin{pmatrix} x_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & x_2 \\ 0 & \cdots & 0 & 0 & x_2 & x_3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & x_2 & \cdots & \cdots & x_{n-1} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-1} \\ a_n \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

As $x_1a_1 = 1$, we have $x_1 \neq 0$ and $a_1 \neq 0$. If $a_n \neq 0$, then from (2), we have $x_2 = x_3 = \cdots = x_{n-1} = 0$. So from (1) we have $x_n = x_1 = 0$. Therefore $a_n = 0$. Similarly we have

$$a_n = a_{n-1} = \cdots = a_2 = 0, \; x_2 = x_3 = \cdots = x_n = 0.$$

This means $f(D_\mu) = E + \mu E_{11} = D_\mu$, which completes the proof.

References

[1] Hua, L. K.: Introduction to Number Theory. Springer-Verlag, New York, pp. 371–382 (1982).

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