Some characterizations of quaternionic space forms

By Toshiaki ADACHI*) and Sadahiro MAEDA**)

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Abstract: In this paper we give characterizations of quaternionic space forms in the class of quaternionic Kähler manifolds in terms of the curvature tensor and the extrinsic shape of geodesics on geodesic spheres.

Key words: Quaternionic Kähler manifolds; curvature tensor; quaternionic space forms; geodesic spheres.

1. Introduction. The aim of this paper is to give two characterizations of quaternionic space forms in the class of quaternionic Kähler manifolds. A quaternionic Kähler structure $\mathcal{J}$ on a Riemannian manifold $(M, \langle\ ,\ \rangle)$ of real dimension $4n$ with Riemannian metric $\langle\ ,\ \rangle$ is a rank 3 vector subbundle of the bundle of endomorphisms of the tangent bundle of M with the following properties:

1) For each point $x \in M$ there is an open neighborhood G of x in M and sections J_1, J_2, J_3 of the restriction $\mathcal{J}|_G$ over G such that
   i) each J_i is an almost Hermitian structure on G, that is,
   $$J_i^2 = -\,\mathrm{id} \quad \text{and} \quad \langle J_iX, Y\rangle + \langle X, J_iY\rangle = 0$$
   for all vector fields X and Y on G.
   ii) $J_iJ_{i+1} = J_{i+2} = -J_{i+1}J_i$ (i mod 3) for $i = 1, 2, 3$.
2) The condition that $\nabla_X J$ is a section of $\mathcal{J}$ holds for each vector field X on M and section J of the bundle $\mathcal{J}$, where ∇ denotes the Riemannian connection of M.

This triple $\{J_1, J_2, J_3\}$ is called a canonical local basis of $\mathcal{J}$. We call a connected quaternionic Kähler manifold $(M, \langle\ ,\ \rangle, \mathcal{J})$ a *quaternionic space form* of *quaternionic sectional curvature* c ($\in \mathbf{R}$) if the Riemannian sectional curvature $\langle R(v, Jv)v, Jv\rangle$ of M is equal to c for each unit vector $v \in TM$ and each unit $J \in \mathcal{J}$. Here we adopt the following signature for Riemannian curvature tensor of M; $R(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z$. Quaternionic space forms are the simplest examples in the class of quaternionic Kähler manifolds. They are locally congruent to either a quaternionic projective space $\mathbf{H}P^n(c)$ of quaternionic sectional curvature $c(>0)$, a quaternionic Euclidean space $\mathbf{H}^n$ or a quaternionic hyperbolic space $\mathbf{H}H^n(c)$ of quaternionic sectional curvature c (<0).

In this paper we characterize quaternionic space forms from two points of view. In section 2, we characterize them by investigating their curvature tensors. In his paper [K] Kosmanek characterized complex space forms in the class of Kähler manifolds (M, J) by the property that the vector $R(u, Ju)u$ is proportional to Ju for all non-null tangent vectors (see also [T]). We here give its quaternionic version.

In section 3, we characterize quaternionic space forms by observing the extrinsic shape of particular geodesics on their geodesic spheres of sufficiently small radius, which is a quaternionic version of our preceding results in [AM].

2. Curvature tensor of quaternionic space forms. For a quaternionic space form of quaternionic sectional curvature c its curvature tensor can be written down as follows(cf. [I]):

$$(2.1)\quad R(X,Y)Z = \frac{c}{4}\Big[\langle X,Z\rangle Y - \langle Y,Z\rangle X + \sum_{i=1}^{3}(\langle Y, J_iZ\rangle J_iX - \langle X, J_iZ\rangle J_iY - 2\langle X, J_iY\rangle J_iZ)\Big],$$

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*) Department of Mathematics, Nagoya Institute of Technology, Gokisocho, Showa-ku, Nagoya, Aichi 466-8555.

**) Department of Mathematics, Shimane University, 1060, Nishi-Kawatsucho, Matsue, Shimane 690-8504.

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where $\{J_1, J_2, J_3\}$ is a canonical local basis of $\mathcal{J}$. This guarantees that $R(u, Ju)u = cJu$, in particular, guarantees that the vector $R(u, Ju)u$ is proportional to Ju for each unit tangent vector u and unit $J \in \mathcal{J}$. In this section we characterize quaternionic space forms by this property. We shall start with the following lemma.

Lemma 1. *The curvature tensor R of a quaternionic Kähler manifold $(M, \langle\ ,\ \rangle, \mathcal{J})$ satisfies the following equality at an arbitrary point $x \in M$:*

$$R(Ju, Jv) = R(u, v)$$

for each unit $J \in \mathcal{J}$ and for each $u, v \in T_xM$ such that v is orthogonal to the quaternionic subspace $\mathcal{J}u$ of T_xM spanned by u.

Proof. For each canonical local basis $\{J_1, J_2, J_3\}$ of quaternionic structure on G, there exist three 1-forms q_1, q_2 and q_3 on G satisfying

$$\nabla_X J_i = q_{i+2}(X)J_{i+1} - q_{i+1}(X)J_{i+2} \quad (i \bmod 3) \tag{2.2}$$

for each vector field X on G and $i = 1, 2, 3$. Using this equality repeatedly, we can verify the following identity for arbitrary vector fields X, Y, Z on G and for $i = 1, 2, 3$:

$$\begin{aligned} R(X, Y)(J_iZ) = J_iR(X, Y)Z &+ \eta_{i,i+1}(X, Y)J_{i+1}Z \\ &+ \eta_{i,i+2}(X, Y)J_{i+2}Z \ (i \bmod 3). \end{aligned} \tag{2.3}$$

Here the differentiable 2-forms $\eta_{i,i+1}$ and $\eta_{i,i+2}$ can be expressed as follows:

$$\begin{cases} \eta_{i,i+1}(X, Y) = q_{i+1}(X)q_i(Y) - q_i(X)q_{i+1}(Y) \\ \qquad -X(q_{i+2}(Y)) + Y(q_{i+2}(X)) + q_{i+2}([X, Y]), \\ \eta_{i,i+2}(X, Y) = q_{i+2}(X)q_i(Y) - q_i(X)q_{i+2}(Y) \\ \qquad +X(q_{i+1}(Y)) - Y(q_{i+1}(X)) - q_{i+1}([X, Y]). \end{cases}$$

Since v is orthogonal to the quaternionic subspace $\mathcal{J}u$, for arbitrary tangent vectors $w, z \in T_xM$ we can see with the aid of (2.3) that

$$\begin{aligned} \langle R(J_iu, J_iv)w, z\rangle &= \langle R(w, z)J_iu, J_iv\rangle \\ &= \langle J_iR(w, z)u, J_iv\rangle \\ &= \langle R(w, z)u, v\rangle = \langle R(u, v)w, z\rangle. \end{aligned}$$

Thus we find $R(J_iu, J_iv) = R(u, v)$ $(i = 1, 2, 3)$. As we can choose a canonical local basis $\{J_1, J_2, J_3\}$ with $J = J_1$ for each unit $J \in \mathcal{J}$, we obtain the conclusion. □

We now show the following characterization of quaternionic space forms:

Theorem 1. *Let $(M, \langle\ ,\ \rangle, \mathcal{J})$ be an $n(\geqq 2)$-dimensional connected quaternionic Kähler manifold. Then M is a quaternionic space form if and only if at an arbitrary point $x \in M$ the vector $R(u, Ju)u$ is proportional to Ju for each tangent vector $u \in T_xM$ and $J \in \mathcal{J}$.*

Proof. We are enough to show the "if" part. First we check

$$\langle R(u, Ju)u, Ju\rangle = \langle R(u, J'u)u, J'u\rangle \tag{2.4}$$

for a tangent vector $u \in T_xM$ and each $J, J' \in \mathcal{J}$ with $\|J\| = \|J'\| = 1$. Indeed, since $\langle (J + J')u, (J - J')u\rangle = 0$, we know by the hypothesis that

$$\begin{aligned} 0 &= \langle R(u, (J + J')u)u, (J - J')u\rangle \\ &= \langle R(u, Ju)u, Ju\rangle - \langle R(u, J'u)u, J'u\rangle. \end{aligned}$$

Next, we check that

$$\langle R(u, Ju)u, Ju\rangle = \langle R(v, Jv)v, Jv\rangle \tag{2.5}$$

for each unit $J \in \mathcal{J}$ and unit tangent vectors $u, v \in T_xM$ such that v is orthogonal to the quaternionic subspace $\mathcal{J}u$ of T_xM. Since $\langle J(u+v), J(u-v)\rangle = 0$, our hypothesis shows

$$\begin{aligned} 0 &= \langle R(u + v, J(u + v))(u + v), J(u - v)\rangle \\ &= \langle R(u, Ju)u, Ju\rangle - \langle R(v, Jv)v, Jv\rangle \\ &\quad - \langle R(u, Jv)u, Jv\rangle + \langle R(v, Ju)v, Ju\rangle. \end{aligned}$$

Here by using Lemma 1 we have

$$\begin{aligned} \langle R(u, Jv)u, Jv\rangle &= \langle R(-Ju, v)u, Jv\rangle \\ &= -\langle R(u, Jv)Ju, v\rangle \\ &= -\langle R(-Ju, v)Ju, v\rangle \\ &= \langle R(v, Ju)v, Ju\rangle. \end{aligned}$$

Hence we obtain (2.5). Combining (2.4) and (2.5), we find at each point $x \in M$ the quaternionic sectional curvature does not depend on the choice of a tangent vector $u \in T_xM$. Therefore the quaternionic version of Schur's theorem tells us that our manifold M is a quaternionic space form (see Theorem 5.3 in [I]). □

Remark. We can relax the condition in Theorem 1 as follows: At an arbitrary point $x \in M$, there is a normal basis $\{I_1, I_2, I_3\}$ of $\mathcal{J}$ such that for each tangent vector $u \in T_xM$ the vector $R(u, I_1u)u$ is proportional to I_1u and the vector $R(u, (I_1 + I_i)u)u$ is orthogonal to $(I_1 - I_i)u$, $i = 2, 3$.

3. Geodesic spheres of quaternionic space forms. In this section we characterize quaternionic space forms by the extrinsic shape of geodesics on geodesic spheres. A smooth curve $\gamma = \gamma(s)$ in a Riemannian manifold M parametrized by its arclength s is called a *circle* of curvature $\kappa\ (\geqq 0)$, if there exists a field of unit vectors Y_s along this curve which satisfies the differential equations: $\nabla_{\dot\gamma}\dot\gamma = \kappa Y_s$, $\nabla_{\dot\gamma} Y_s = -\kappa\dot\gamma$, where κ is a constant and $\nabla_{\dot\gamma}$ denotes the covariant differentiation along γ with respect to the Riemannian connection ∇ of M. A circle of null curvature is nothing but a geodesic.

The following lemma is based on the Taylor expansion for the second fundamental form of geodesic spheres in an arbitrary Riemannian manifold M, which is due to Chen and Vanheche [CV].

Lemma 2. *For non-zero tangent vectors $v, w \in T_xM$ at a point $x \in M$, we choose a unit tangent vector $u \in T_xM$ orthogonal to both v and w. We respectively denote by $v_r, w_r \in T_{\exp_x(ru)}M$ the parallel displacements of v and w along the geodesic segment $\exp_m(su)$, $0 \leqq s \leqq r$. Then for sufficiently small r we have*

$$\text{(3.1)}\quad \langle A_{m,r}v_r, w_r\rangle = \frac{1}{r}\langle v, w\rangle - \frac{r}{3}\langle R(u,v)w, u\rangle + O(r^2),$$

where $A_{m,r}$ denotes the shape operator of the geodesic sphere $S_x(r)$ of radius r centered at x.

Let N be a real hypersurface of a quaternionic Kähler manifold M and $\mathcal{N}$ a unit local normal vector field of N in M. We denote by $\mathcal{D}$ the maximal subbundle of TN which is invariant by the quaternionic Kähler structure $\mathcal{J}$: At a point $y \in N\ (\subset M)$ the subspace $\mathcal{D}_y$ is the maximal subspace of T_yN with the property that $Jv \in \mathcal{D}_y$ for each $v \in \mathcal{D}_y$ and $J \in \mathcal{J}_y$. Let $\mathcal{D}^\perp$ denote the orthogonal complement of $\mathcal{D}$ in TN. It is a rank 3 vector subbundle of TN. By using a canonical local basis $\{J_1, J_2, J_3\}$ of $\mathcal{J}$ over an open subset G of M containing y, we find that $\mathcal{D}_y^\perp$ is the real linear subspace of T_yN spanned by $J_1\mathcal{N}, J_2\mathcal{N}, J_3\mathcal{N}$. On an open set $G \cap N$ we set $\xi_i = -J_i\mathcal{N}$ and define $\phi_i : TN \to TN$ by $\phi_i = \pi \circ J_i|_{TN}$ for $i = 1, 2, 3$, where $\pi : TM|_N \to TN$ is the canonical projection. Then the following identities hold on $G \cap N$ for $i = 1, 2, 3$:

$$\phi_i\xi_i = 0,\ \phi_i\xi_{i+1} = \xi_{i+2},\ \phi_i\xi_{i+2} = -\xi_{i+1},$$
$$\phi_i \circ \phi_{i+1}|_{\mathcal{D}} = \phi_{i+2}|_{\mathcal{D}} = -\phi_{i+1} \circ \phi_i|_{\mathcal{D}} \quad (i \bmod 3).$$

From now on, we pay attention on geodesics γ on geodesic spheres with the initial vector $\dot\gamma(0) \in \mathcal{D}^\perp_{\gamma(0)}$.

In a quaternionic Euclidean space $\mathbf{H}^n$, a geodesic sphere $S_x(r)$ is nothing but a $(4n-1)$-dimensional standard sphere of curvature $1/r^2$ as a totally umbilic hypersurface of $\mathbf{R}^{4n} (= \mathbf{H}^n)$. In particular, every geodesic on $S_x(r)$ is a circle of curvature $1/r$ in $\mathbf{H}^n$. On the other hand, each nonflat quaternionic space form $M(c)$ $(= \mathbf{H}P^n(c)$ or $\mathbf{H}H^n(c))$ does not admit a real hypersurface N all of whose geodesics are circles in the ambient space $M(c)$, because there exist no totally umbilic real hypersurfaces of $M(c)$. However $S_x(r)$ is the simplest example of real hypersurfaces in $M(c)$, $c \neq 0$. For latter use, we here summarize some of basic properties on the shape operator A of a geodesic sphere in a nonflat quaternionic space form.

Lemma 3 (cf. [P])**.** *Every geodesic sphere $S_x(r)$ in a nonflat quaternionic space form M satisfies the following*:

(1) *The structure tensor ϕ_i and the shape operator A of $S_x(r)$ are commutative*: $\phi_i A = A\phi_i$ $(i = 1, 2, 3)$.

(2) *The shape operator A of $S_x(r)$, $0 < r < \pi/2$ in $\mathbf{H}P^n(4)$ satisfies the following at each point $y \in S_x(r)$*:

$$Au = \cot r \cdot u \quad \textit{for all} \quad u \in \mathcal{D}_y$$

and

$$A\xi = 2\cot 2r \cdot \xi \quad \textit{for all} \quad \xi \in \mathcal{D}_y^\perp,$$

and the shape operator A of $S_x(r)$, $0 < r < \infty$ in $\mathbf{H}H^n(-4)$ satisfies the following at each point $y \in S_x(r)$:

$$Au = \coth r \cdot u \quad \textit{for all} \quad u \in \mathcal{D}_y$$

and

$$A\xi = 2\coth 2r \cdot \xi \quad \textit{for all} \quad \xi \in \mathcal{D}_y^\perp.$$

(3) *The covariant derivative of the shape operator A of $N = S_x(r)$ satisfies*

$$({}^N\nabla_X A)Y = \mp\sum_{i=1}^{3}\{\langle\phi_i X, Y\rangle\xi_i + \langle\xi_i, Y\rangle\phi_i X\},$$

where the double sign depends on the case that either the ambient space is $\mathbf{H}P^n(4)$ or $\mathbf{H}H^n(-4)$.

We are now in a position to prove the following.

Theorem 2. *Let M be an $n\ (\geqq 2)$-dimensional connected quaternionic Kähler manifold. Then the following conditions are equivalent.*

(1) *M is a quaternionic space form.*

(2) *Consider an arbitrary geodesic sphere $S_x(r)$ of sufficiently small radius r centered at an arbi-*

trary point $x \in M$. Every geodesic $\gamma = \gamma(s)$ with $\gamma(0) = y$ and $\dot\gamma(0) \in \mathcal{D}_y^\perp$ for an arbitrary fixed point $y \in S_x(r)$ is a circle of positive curvature in the ambient manifold M.

Proof. (1) $\Longrightarrow$ (2). It suffices to check the case where the ambient manifold M is one of $\mathbf{H}P^n(4)$ and $\mathbf{H}H^n(-4)$. First we consider a geodesic sphere $S_x(r)$ in $\mathbf{H}P^n(4)$. We denote by ${}^N\nabla$ and ∇ the Riemannian connections of $N = S_x(r)$ and $\mathbf{H}P^n(4)$, respectively. Let $\gamma = \gamma(s)$ be a geodesic on $S_x(r)$ with the initial condition that $\gamma(0) = y \in N$ and $\dot\gamma(0) \in \mathcal{D}_y^\perp$. We shall show that $A\dot\gamma(s) = 2\cot 2r \cdot \dot\gamma(s)$ for every s. It follows from the third assertion of Lemma 3 that

$$\begin{aligned}
&{}^N\nabla_{\dot\gamma}\|A\dot\gamma(s) - 2\cot 2r \cdot \dot\gamma(s)\|^2 \\
&= {}^N\nabla_{\dot\gamma}\langle A\dot\gamma, A\dot\gamma\rangle - 4\cot 2r\,{}^N\nabla_{\dot\gamma}\langle A\dot\gamma, \dot\gamma\rangle \\
&= 2\langle({}^N\nabla_{\dot\gamma}A)\dot\gamma, A\dot\gamma\rangle - 4\cot 2r\langle({}^N\nabla_{\dot\gamma}A)\dot\gamma, \dot\gamma\rangle \\
&= -2\Big\langle\sum_{i=1}^{3}\{\langle\phi_i\dot\gamma, \dot\gamma\rangle\xi_i + \langle\xi_i, \dot\gamma\rangle\phi_i\dot\gamma\}, A\dot\gamma - 2\cot 2r \cdot \dot\gamma\Big\rangle \\
&= -2\sum_{i=1}^{3}\langle\xi_i, \dot\gamma\rangle\langle\phi_i\dot\gamma, A\dot\gamma\rangle.
\end{aligned}$$

Since $A\phi_i = \phi_i A$, we find $\langle\phi_i\dot\gamma, A\dot\gamma\rangle = \langle A\phi_i\dot\gamma, \dot\gamma\rangle = -\langle A\dot\gamma, \phi_i\dot\gamma\rangle$, which leads us to $\langle\phi_i\dot\gamma, A\dot\gamma\rangle = 0$, and hence to ${}^N\nabla_{\dot\gamma}\|A\dot\gamma(s) - 2\cot 2r \cdot \dot\gamma(s)\|^2 = 0$. As we have $A\dot\gamma(0) = 2\cot 2r \cdot \dot\gamma(0)$ by the initial condition, we obtain the desirable equality $A\dot\gamma(s) = 2\cot 2r \cdot \dot\gamma(s)$ for every s. By use of the formulae of Gauss and Weingarten

$$\nabla_X Z = {}^N\nabla_X Z + \langle AX, Z\rangle\mathcal{N} \quad \text{and} \quad \nabla_X\mathcal{N} = -AX,$$

we can see that

$$\nabla_{\dot\gamma}\dot\gamma = 2\cot 2r \cdot \mathcal{N} \quad \text{and} \quad \nabla_{\dot\gamma}\mathcal{N} = -2\cot 2r \cdot \dot\gamma,$$

which means that the extrinsic shape of the geodesic γ is a circle of curvature $2\cot 2r$ in $\mathbf{H}P^n(4)$.

For geodesics on geodesic spheres in $\mathbf{H}H^n(-4)$ we can get the similar result by the same argument.

(1) $\Longleftarrow$ (2). Let $\gamma = \gamma(s)$ be a geodesic on $S_x(r)$ with the initial condition that $\gamma(0) = y$ and $\dot\gamma(0) = \xi \in \mathcal{D}_y^\perp$, where ξ is an arbitrary fixed unit vector. It follows from the formulae of Gauss and Weingarten that

$$\begin{aligned}
(3.2)\quad &\nabla_{\dot\gamma}(\nabla_{\dot\gamma}\dot\gamma) \\
&= \langle({}^N\nabla_{\dot\gamma}A_{m,r})\dot\gamma, \dot\gamma\rangle\mathcal{N} - \langle A_{m,r}\dot\gamma, \dot\gamma\rangle A_{m,r}\dot\gamma.
\end{aligned}$$

On the other hand, since γ is a circle in M by the hypothesis, there exists a positive constant κ_γ satisfying that

$$(3.3)\qquad \nabla_{\dot\gamma}(\nabla_{\dot\gamma}\dot\gamma) = -\kappa_\gamma^2\dot\gamma.$$

Comparing the tangential components of (3.2) and (3.3), we find the following:

$$\langle A_{m,r}\dot\gamma, \dot\gamma\rangle A_{m,r}\dot\gamma = \kappa_\gamma^2\dot\gamma.$$

As $\kappa_\gamma \neq 0$, at $s = 0$ we have for every unit vector $\xi \in \mathcal{D}_y^\perp$

$$A_{m,r}\xi = \kappa_\xi\xi \quad \text{or} \quad A_{m,r}\xi = -\kappa_\xi\xi$$

for some positive κ_ξ. Since we have

$$\kappa_{\xi+\xi'}(\xi + \xi') = A_{m,r}(\xi + \xi') = \kappa_\xi\xi + \kappa_{\xi'}\xi',$$

we find a constant κ_y satisfying that

$$A_{m,r}\xi = \kappa_y\xi \quad \text{for all} \quad \xi \in \mathcal{D}_y^\perp.$$

Here, we employ Lemma 2. Let $u \in T_xM$ be any fixed unit vector at an arbitrary point $x \in M$ and J be any element in $\mathcal{J} \setminus \{0\}$. We choose a tangent vector $w \in T_xM$ orthogonal to both u and Ju and set $v = Ju$. Since u_r is a normal vector of the geodesic sphere $S_x(r)$ in M at $y = \exp_x(ru)$, we know that $v_r \in \mathcal{D}_y^\perp$, so that $A_{m,r}v_r = \kappa_y v_r$. Hence the expansion (3.1) shows that the curvature tensor R of M satisfies

$$\langle R(u, Ju)w, u\rangle = 0.$$

Therefore we can see that $R(u, Ju)u$ is proportional to Ju for every $u \in T_xM$ and for every $J \in \mathcal{J}$ at each point $x \in M$, so that M is a quaternionic space form by Theorem 1. □

The following is an improvement of Theorem 2.

Theorem 3. *Let M be an n ($\geqq 2$)-dimensional connected quaternionic Kähler manifold. Then the following conditions are equivalent.*

(1) *M is a quaternionic space form.*
(2) *Consider an arbitrary geodesic sphere $S_x(r)$ of sufficiently small radius r centered at an arbitrary point $x \in M$. At each point $y \in S_x(r)$ there exists an orthonormal basis $\{\eta_1, \eta_2, \eta_3\}$ of $\mathcal{D}_y^\perp$ such that all geodesics on $S_x(r)$ through y in the direction $\eta_i + \eta_j$ ($i, j = 1, 2, 3$) are circles of positive curvature in the ambient manifold M.*

Proof. We are enough to show that the condition (2) implies that M is a quaternionic space form. Let $\gamma_i = \gamma_i(s)$ ($i = 1, 2, 3$) be geodesics on $S_x(r)$ with $\gamma_i(0) = y$ and $\dot\gamma_i(0) = \eta_i$ and $\sigma_j = \sigma_j(s)$ ($j = 2, 3$) be geodesics on $S_x(r)$ through y in the direction $\eta_1 + \eta_j$. Then the same discussion as in the proof of Theorem

2 yields

$$\langle A_{m,r}\eta_i, \eta_i\rangle A_{m,r}\eta_i = \kappa_i^2\eta_i, \quad i = 1, 2, 3$$
$$\langle A_{m,r}(\eta_1 + \eta_j), (\eta_1 + \eta_j)\rangle A_{m,r}(\eta_1 + \eta_j)$$
$$= \lambda_j^2(\eta_1 + \eta_j), \quad j = 2, 3$$

for some positive constants κ_i and λ_j. Hence we find $A_{m,r}\eta_i = \pm\kappa_i\eta_i$, and $A_{m,r}(\eta_1 + \eta_j) = \pm\lambda_j(\eta_1 + \eta_j)$. Since $\langle \eta_1 + \eta_j, \eta_1 - \eta_j\rangle = 0$, $j = 2, 3$, we find

$$\langle A_{m,r}(\eta_1 + \eta_j), \eta_1 - \eta_j\rangle = 0,$$

so that

$$\langle A_{m,r}\eta_1, \eta_1\rangle = \langle A_{m,r}\eta_j, \eta_j\rangle,$$

because $A_{m,r}$ is symmetric. Thus there exists a constant κ_y with

$$A_{m,r}\xi = \kappa_y\xi \quad \text{for all} \quad \xi \in \mathcal{D}_y^{\perp}.$$

By the proof of Theorem 2 we get our conclusion. □

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