On Greenberg's conjecture on a certain real quadratic field

By Yasushi Mizusawa

Department of Information and Computer Science, School of Science and Engineering, Waseda University, 3-4-1, Okubo, Shinjuku-ku, Tokyo 169-8555

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Abstract: In this paper, we prove by using Ichimura-Sumida criterion the vanishing of the Iwasawa λ_3-invariant of the real quadratic field $\mathbf{Q}(\sqrt{39345017})$. We use also the fact that the field has the infinite 3-class field tower.

Key words: Iwasawa invariant; class field tower; real quadratic field.

1. Introduction. Let p be an odd prime number and k a real quadratic field. We consider the following two sequences of p-extensions of k: Let k_∞ be the cyclotomic $\mathbf{Z}_p$-extension of k, which is regarded as a sequence of the unique subfield k_n of degree p^n over k for $n \geq 0$. In the case $k = \mathbf{Q}(\sqrt{39345017})$ which will be the main theme of the present paper, k_∞/k is totally ramified at all primes over $p = 3$, and unramified outside p. Let $\lambda_p(k)$ be the Iwasawa λ-invariant of k_∞ over k. Greenberg's conjecture for k_∞ asserts that $\lambda_p(k) = 0$, in other words, p-Hilbert class field of k_∞ is finite over k_∞. In [1] and [2], Ichimura and Sumida established a remarkable criterion which is based on a deep investigation of properties of cyclotomic units and Iwasawa polynomials. By this criterion, it is confirmed that $\lambda_3(k) = 0$ for all $k = \mathbf{Q}(\sqrt{m})$ with $1 < m < 10^4$.

As the other sequence, we consider the p-class field tower of k denoted by $k^{(\infty)}$, defined as follows. Let $k^{(0)} = k$ and $k^{(n+1)}$ be the p-Hilbert class field of $k^{(n)}$ for $n \geq 0$, then we have

$$k = k^{(0)} \subseteq k^{(1)} \subseteq k^{(2)} \subseteq \cdots \subseteq k^{(\infty)} = \bigcup_{n \geq 0} k^{(n)}.$$

The field $k^{(\infty)}$ is the maximal unramified p-extension of k. The p-class field tower of k is called finite if $k^{(\infty)}$ is a finite extension of k and infinite otherwise. Let $Cl(k)$ be the ideal class group of k. In quadratic case, it is known that if *p-rank of* $Cl(k) \geq 3$ then k has infinite p-class field tower. We know that $k = \mathbf{Q}(\sqrt{39345017})$ has class number 27 and 3-*rank of* $Cl(k) = 3$, hence k has infinite 3-class field tower (cf. Schoof [4]).

If k has the infinite p-class field tower and $\lambda_p(k) = 0$, the maximal unramified p-extension of k_∞ is also infinite over k_∞, but the maximal unramified abelian p-extension of k_∞ is finite over k_∞. In [3], Ozaki constructed infinitely many cyclotomic $\mathbf{Z}_2$-extensions k_∞ of real quadratic fields k such that $\lambda_2(k) = 0$ and k_n has the infinite 2-class field tower for sufficiently large $n \geq 0$. The purpose of this paper is to prove the vanishing of the Iwasawa λ_3-invariant of the real quadratic field $k = \mathbf{Q}(\sqrt{39345017})$, applying Ichimura-Sumida criterion to k.

2. Ichimura-Sumida criterion. We denote by f the conductor of a real quadratic field k. Let χ be a $\mathbf{Q}_p$-valued non-trivial Dirichlet character associated to k and ω a Teichmüller character $\mathbf{Z}/p\mathbf{Z} \to \mathbf{Z}_p$. In this section, we assume $\chi\omega^{-1}(p) \neq 1$ and that p remains prime in k. We fix a topological generator γ of $\Gamma = \mathrm{Gal}(k_\infty/k)$ and an isomorphism $\mathbf{Z}_p[[\Gamma]] \simeq \mathbf{Z}_p[[T]] : \gamma \mapsto 1+T$, and take a generator σ of $\Delta = \mathrm{Gal}(k_\infty/\mathbf{Q}_\infty)$. Let $g_\chi(T)$ be a power series in $\Lambda = \mathbf{Z}_p[[T]]$ related to the p-adic L-function $L_p(s,\chi)$ such that

$$g_\chi((1+fp)^{1-s} - 1) = L_p(s,\chi).$$

By p-adic Weierstrass preparation theorem, we obtain a distinguished polynomial $P_\chi(T)$ of $g_\chi(T)$, called Iwasawa polynomial. It is known that $\lambda_p(k) \leq \deg P_\chi(T)$ by Iwasawa main conjecture, so we consider the case $\deg P_\chi(T) > 0$. Let $P_i(T)$ $(1 \leq i \leq r)$ be all irreducible factors of $P_\chi(T)$ in $\mathbf{Z}_p[T]$, and put $\omega_n(T) = (1+T)^{p^n} - 1$ $(n \geq 0)$. As Leopoldt's conjecture holds for $(k_n,\, p)$, the $\mathbf{Z}_p$-modules $\Lambda/(P_i, \omega_n)$ are finite. We denote by $p^{a_{i,n}}$ the exponent of $\Lambda/(P_i, \omega_n)$, and let $X_{i,n}(T)$ be the unique polynomial in $\mathbf{Z}_p[T]$ satisfying $X_{i,n}P_i \equiv p^{a_{i,n}} \pmod{\omega_n}$

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and $\deg X_{i,n} < p^n$. We take a polynomial $Y_{i,n}(T)$ in $\mathbf{Z}[T]$ such that $Y_{i,n}(T) \equiv X_{i,n}(T) \pmod{p^{a_{i,n}}}$.

Let z be an integer satisfying $2z \equiv 1 \pmod{p^{a_{i,n}}}$, and take an element $\mathbf{e}_{i,n} = z(1-\sigma)$ in $\mathbf{Z}[\Delta]$. We take the cyclotomic unit

$$c_{i,n} = \{\mathrm{Norm}_{\mathbf{Q}(\zeta_{fp^{n+1}})/k_n}(1-\zeta_{fp^{n+1}})^{p^2-1}\}^{\mathbf{e}_{i,n}}.$$

Now, the Ichimura-Sumida criterion states as follows.

Theorem 1 (cf. Theorem in [1] or Corollary 1 in [2]). *Under the above setting, we have $\lambda_p(k)=0$ if and only if for any i $(1 \le i \le r)$, the condition*

$$(H_{i,n}) \qquad (c_{i,n})^{Y_{i,n}(\gamma-1)} \notin (k_n^{\times})^{p^{a_{i,n}}}$$

holds for some $n \ge 0$.

We note that the condition $(H_{i,n})$ implies $(H_{i,n+1})$ (cf. Lemma 1 in [2]).

3. Computational result. Let $k = \mathbf{Q}(\sqrt{39345017})$. The field k has the prime conductor $f = 39345017$, and 3 remains prime in k. Then we can apply the Theorem 1 to k with $p=3$. In this section, we prove vanishing of the Iwasawa λ_3-invariant $\lambda_3(k)$.

By calculating $g_\chi(T)$ modulo $(\omega_8, 3^9)$, we have

$$P_\chi(T) \equiv T^6 + 2175T^5 + 1737T^4 + 1596T^3 + 621T^2 + 936T + 1917 \pmod{3^7}.$$

By Hensel's lemma, it is decomposed to $P_\chi(T) = P_1(T)P_2(T)P_3(T)$ with

$$P_1(T) \equiv T + 219 \pmod{3^5},$$
$$P_2(T) \equiv T^2 + 81T + 222 \pmod{3^5},$$
$$P_3(T) \equiv T^3 + 12T^2 + 21T + 66 \pmod{3^4}.$$

For P_1, we have $a_{1,3} = 4$ and obtain $Y_{1,3}(T)$ by Euclidean algorithm. But $(c_{1,3})^{Y_{1,3}(\gamma-1)}$ is 3^4-th power in k_3, so we have to work in k_4. There is a surjective homomorphism $\Lambda/(\omega_4/\omega_3, P_1) \to (\omega_3, P_1)/(\omega_4, P_1)$ as Λ-modules, and we see that the exponent of $\Lambda/(\omega_4/\omega_3, P_1)$ is 3. We have $a_{1,4} = 5$ and obtain $Y_{1,4}(T) \in \mathbf{Z}[T]$ such that $\omega_4(T)u \equiv (T+219)Y_{1,4}(T) + 3^5 \pmod{3^6}$ for some $u \in \mathbf{Z} \cap \mathbf{Z}_3^{\times}$. Fortunately we see that $(c_{1,4})^{Y_{1,4}(\gamma-1)}$ is 3^4-th power but not 3^5-th power in k_4. For P_2 and P_3, we have $a_{2,2} = 3$ and $a_{3,2} = 2$. We also see that $(c_{2,2})^{Y_{2,2}(\gamma-1)}$ is 9-th power but not 27-th power in k_2 and $(c_{3,2})^{Y_{3,2}(\gamma-1)}$ is cube but not 9-th power in k_2. Hence we can conclude that $\lambda_3(k) = 0$ and obtain the following result by the Theorem 1.

Theorem 2. *The Iwasawa λ_3-invariant of a real quadratic field $\mathbf{Q}(\sqrt{39345017})$ which has infinite 3-class field tower is zero.*

Our computations were carried out by using excellent calculation software UBASIC86 Ver. 8.8 which is available by ftp at `ftp://rkmath.rikkyo.ac.jp/`.

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