Note on steady solutions of the Eguchi-Oki-Matsumura equation

By Takao Hanada,*) Naoyuki Ishimura,**) and MasaAki Nakamura***)

(Communicated by Heisuke Hironaka, M. J. A., Nov. 13, 2000)

Abstract: Eguchi-Oki-Matsumura (EOM) equations are introduced to model theoretically the kinetics of ordering which accompanies phase separation in some binary alloys. Numerical analysis shows that EOM equations admit several steady states. Employing the variational structure associated with the free energy, we prove that there really exist non-trivial steady solutions for EOM equations.

Key words: Phase separation; Eguchi-Oki-Matsumura equations; steady solutions; variational structure.

1. Introduction. In thermodynamics, the phenomena of phase separation observed in certain alloys have been an attractive subject for researches. Eguchi-Oki-Matsumura [1], in an attempt of theoretically investigating such kinetics, introduced a system of equations, which will be referred to as EOM equations hereafter. This motion law is derived from the first principles of thermodynamics of irreversible process under appropriate assumptions on the free energy, and consists of coupled two variables; one is the local concentration and the other is the local degree of order. After performing a suitable scaling of parameters, EOM equations in one-space dimension are expressed as follows.

$$(1)\qquad \begin{cases} u_t = -\varepsilon^2 u_{xxxx} + ((a+v^2)u)_{xx} \\ \qquad\qquad \text{in } x \in (0,1), \quad t>0 \\ v_t = v_{xx} + (b - u^2 - v^2)v \\ \qquad\qquad \text{in } x \in (0,1), \quad t>0 \\ u_x = u_{xxx} = v_x = 0 \\ \qquad\qquad \text{at } x = 0,1, \quad t>0, \end{cases}$$

where $u = u(x,t)$ and $v = v(x,t)$ denote unknown functions related to the local concentration and the local degree of order, respectively. The total concentration of u is conserved under the evolution of (1). Namely we have

$$\int_0^1 u(x,t)\,dx = m,$$

where m is a constant.

ε, a are positive constants depending on the temperature, and $b \in \mathbf{R}$ is a constant which increase from negative to positive as the temperature crosses downward the critical one. We focus our attention, however, on the case of positive b, since the negative b turns out to enjoy rather trivial behaviors.

As a special case of EOM equations, we remark that if $v \equiv 0$ then (1) reduces to

$$\begin{cases} u_t = -\varepsilon^2 u_{xxxx} + a u_{xx} \\ \qquad\qquad \text{in } x \in (0,1), \quad t>0 \\ u_x = u_{xxx} = 0 \\ \qquad\qquad \text{at } x = 0,1, \quad t>0 \\ \int_0^1 u\,dx = m. \end{cases}$$

This is the famous Cahn-Hilliard equation in its simplest form. While if we put $u \equiv m$ in (1) then we recover

$$\begin{cases} v_t = v_{xx} + (b - m^2 - v^2)v \\ \qquad\qquad \text{in } x \in (0,1), \quad t>0 \\ v_x = 0 \quad \text{at } x = 0,1, \quad t>0, \end{cases}$$

which is deduced from the Ginzburg-Landau theory for superconductivity.

In this note, we are concerned with steady solutions of (1). To be precise, we want to seek for

1991 Mathematics Subject Classification. Primary 35J50; Secondary 80A30.

*) Department of Mathematics, Chiba Institute of Technology, 2-1-1, Shibazono, Narashino, Chiba 275-0023.
**) Department of Mathematics, Faculty of Economics, Hitotsubashi University, 2-1, Naka, Kunitachi, Tokyo 186-8601.
***) College of Science and Technology, Nihon University, 1-8-14, Kanda-Surugadai, Chiyoda-ku, Tokyo 101-8308.

solutions $u = u(x)$ and $v = v(x)$ which verify

$$(2)\quad \begin{cases} -\varepsilon^2 u_{xxxx} + ((a + v^2)u)_{xx} = 0 \\ \qquad \text{in } x \in (0,1) \\ v_{xx} + (b - u^2 - v^2)v = 0 \\ \qquad \text{in } x \in (0,1) \\ u_x = u_{xxx} = v_x = 0 \\ \qquad \text{at } x = 0, 1 \\ \displaystyle\int_0^1 u\,dx = m. \end{cases}$$

As is easily seen, (2) always has a solution $u \equiv m$ and $v \equiv 0$. If $b \le 0$ then the maximum principle implies that this is the only solution to (2). If $b > m^2$, (2) has another solution $u \equiv m$ and $v \equiv \pm\sqrt{b - m^2}$. We call these solutions as trivial solutions of EOM equations.

Our intention is now to discuss whether there exists other solution to (2) or not. Indeed, numerical computation on (1) indicates that there are various steady solutions different from trivial solutions [3], which will be called non-trivial. The aim of the present note is to confirm analytically this numerical observation. Our main results read as follows.

Theorem 1. *For all large $b \gg m^2$, there exists at least one non-trivial steady solution for* EOM *equations.*

The largeness of b and m^2 stated in Theorem 1 can be computed explicitly, though we do not need it.

The proof of Theorem 1 is carried out from the variational point of view in the next section, where other properties of steady solutions are also exhibited. For the study on the evolution system (1) itself, we refer to [2] for instance.

2. Variational structure for steady solutions. Although (2) is a system of equations of fourth-order, it has a second-order variational structure; the solution to (2) is given by the critical point of a functional

$$F[u,v] := \int_0^1 \Big(\frac{\varepsilon^2}{2} u_x^2 + \frac{1}{2} v_x^2 + \frac{a}{2} u^2 + \frac{1}{4} v^4 - \frac{b}{2} v^2 + \frac{1}{2} u^2 v^2 \Big)\,dx$$

among the function spaces (referred to as admissible functions)

$$\mathcal{A} := \Big\{ (u,v) \in (H^1(0,1))^2 \;\Big|\; u_x = v_x = 0 \text{ at } x = 0, 1 \text{ and } \int_0^1 u\,dx = m \Big\}.$$

We immediately obtain

$$F[m,0] = \frac{a}{2} m^2,$$
$$F[m, \pm\sqrt{b - m^2}] = \frac{a}{2} m^2 - \frac{1}{4}(b - m^2)^2 \quad \text{if } b > m^2.$$

Since F is bounded below on $\mathcal{A}$, our task of finding a non-trivial steady solution is to choose a test function $(u,v) \in \mathcal{A}$ such that

$$(3)\qquad F[u,v] < \frac{a}{2} m^2 - \frac{1}{4}(b - m^2)^2.$$

The minimization procedure then yields the solution we want.

To accomplish this, we take

$$u(x) = m - \delta \cos \pi x$$
$$v(x) = \pm\sqrt{b - (m - \delta \cos \pi x)^2},$$

where $\delta > 0$ is a parameter and we assign $b > (m + \delta)^2$. Clearly $(u,v) \in \mathcal{A}$ and we compute

$$F[u,v] = \frac{a}{2} m^2 - \frac{1}{4}(b - m^2)^2 + (\varepsilon^2 \pi^2 + a)\frac{\delta^2}{4} + \frac{\delta^2 \pi^2}{2} \int_0^1 \frac{(m - \delta \cos \pi x)^2 \sin^2 \pi x}{b - (m - \delta \cos \pi x)^2}\,dx - \frac{m^2 \delta^2}{2} - \frac{3}{32}\delta^4 + \frac{\delta^2}{4}(b - m^2).$$

If we further set $b = 2m^2$ and $\delta = m/4$, then we infer that

$$F[u,v] = F[m, \pm\sqrt{b - m^2}] - \frac{131}{2^{13}} m^4 + \left(\frac{\varepsilon^2 \pi^2 + a}{64} + \frac{\pi^2}{32} \int_0^1 \frac{(4 - \cos \pi x)^2 \sin^2 \pi x}{32 - (4 - \cos \pi x)^2}\,dx \right) m^2,$$

from which we conclude that (3) holds, taking m larger if necessary. The proof of Theorem 1 is completed.

We end the current short article with establishing additional results concerning local minimizers for F. We recall that $(u_0, v_0) \in \mathcal{A}$ is a local minimizer if there is a neighborhood $\mathcal{U}$ of (u_0, v_0) in $\mathcal{A}$ such that $F[u_0, v_0] \le F[u,v]$ for all $(u,v) \in \mathcal{U}$; (u_0, v_0) is a steady solution of EOM equations and the second variation of F at (u_0, v_0) is non-negative. A standard argument now leads to

Theorem 2. *All local minimizers for F are monotone functions.*

Of course not every steady solution is attained by a local minimizer; there is a strong numerical evidence that exist steady solutions which are not local minimizers. These steady solutions are not necessarily monotone [3]. Our test functions, on the other hand, are monotone and they may be in a neighborhood of the global minimizer for F.

Acknowledgements. We are grateful to Prof. Hiroshi Fujita for his interest in this research. This work is partially supported by Grants-in-Aids for Scientific Research (No.10555023), from Japanese Ministry of Education, Science, Sports and Culture.

References

[1] Eguchi, T., Oki, K., and Matsumura, S.: Kinetics of ordering with phase separation. Mat. Res. Soc. Symp. Proc., **21**, Elsevier, Amsterdam-New York, pp. 589–594 (1984).

[2] Hanada, T., Nakamura, M. A., and Shima, C.: On Eguchi-Oki-Matsumura equations. Advances in Numerical Mathematics; Proceedings of the Fourth Japan-China Joint Seminar on Numerical Mathematics (eds. Kawarada, H., Nakamura, M. A., and Shi, Z.-C.). Gakuto, Tokyo, pp. 213–222 (1999).

[3] Hanada, T., and Nakamura, M. A.: Numerical computation of stationary states for phase separation phenomena based on the Eguchi-Oki-Matsumura model (preprint).