Milnor numbers and classes of local complete intersections

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1. Introduction. Let V be an n-dimensional compact complex subvariety of a complex manifold M. When V is non-singular, the Chern classes of the complex tangent bundle TV are well-defined cohomology classes in $H^*(V;\mathbf{Z})$. We denote by $c_*(V)$ their image by the Poincaré isomorphism

$$P_V : H^{2(n-i)}(V;\mathbf{Z}) \xrightarrow{\frown[V]} H_{2i}(V;\mathbf{Z}),$$

cap-product by the fundamental class $[V]$ of V. When V is singular there is no more Chern cohomology classes, but there are several theories generalizing homology classes $c_*(V)$. For instance, the Chern-Schwartz-MacPherson classes $c_*^{SM}(V)$ ([16], [17], [10], [3]) and the Fulton-Johnson classes $c_*^{FJ}(V)$ [5] are two different theories which coincide with $c_*(V)$ when V is non-singular. Our main purpose is to compare the Chern-Schwartz-MacPherson and the Fulton-Johnson classes when V is a local complete intersection. In this paper, we give a presentation of the main results; the complete proofs will be published elsewhere (see [4]).

On one hand, M. H. Schwartz defined actually classes in $H^*(M, M-V;\mathbf{Z})$ ([16], 1965). Let us denote by m the complex dimension of M. It is proved in [3](1979) that Schwartz classes are mapped by the Alexander duality

$$H^{2(m-i)}(M, M-V;\mathbf{Z}) \longrightarrow H_{2i}(V;\mathbf{Z})$$

onto the classes defined by MacPherson ([10], 1974).

We restrict ourselves to the case of a local complete intersection V defined by a holomorphic section of a vector bundle. We consider a holomorphic vector bundle $E \to M$ of rank $k = m-n$, and a holomorphic section s generically transverse to the zero section, such that V is the zero set $s^{-1}(0)$. In this case, the virtual classes of V are defined in [4] as the Chern classes $c^*_{vir}(V) \in H^*(V;\mathbf{Z})$ of the "virtual tangent bundle" $[TM-E]|_V$ (in the complex K-theory $\tilde{K}(V)$). The virtual classes $c^*_{vir}(V)$ coincide with the usual Chern classes if V is non-singular and their images by the Poincaré duality (no more an isomorphism), denoted by $c_*^{vir}(V)$, coincide with the Fulton-Johnson classes $c_*^{FJ}(V)$.

In order to compare the Schwartz-MacPherson and the Fulton-Johnson classes of a local complete intersection, we have to study the difference $c_*^{vir}(V) - c_*^{SM}(V)$. This difference localizes near the singular part $\mathrm{Sing}(V)$ of V: more precisely, if we denote by $(S_\alpha)_\alpha$ the family of connected components of $\mathrm{Sing}(V)$, there are well defined elements $\mu_*(V, S_\alpha)$ in $H_*(S_\alpha;\mathbf{Z})$, called "the (homological) Milnor classes" of V at S_α, such that we get the

Theorem A. *We have,*

$$c_*^{vir}(V) - c_*^{SM}(V) = (-1)^n \sum_\alpha (i_\alpha)_*\big(\mu_*(V, S_\alpha)\big),$$

where $(i_\alpha)_* : H_*(S_\alpha) \to H_*(V)$ denotes the natural map arising from the inclusion $S_\alpha \subset V$.

The Milnor number is well defined by Milnor [11], for hypersurfaces with isolated singular points, by Hamm [7] and Lê [8] for local complete intersections still with isolated singular points, and by Parusiński [12] for hypersurfaces with any compact singular set. The following theorem justifies the terminology "Milnor class" that we use.

Theorem B. *$\mu_0(V, S_\alpha)$ is equal to the Milnor number of V at S_α in $H_0(S_\alpha) \cong \mathbf{Z}$, in all situations where this number has been already defined.*

Such a theory for Milnor classes in homology has also been suggested by Yokura [21], and given in the case of complex compact hypersurfaces by Aluffi [1] and Parusiński-Pragacz [14].

For $r \geq 1$, we explain how to compute the Milnor class $\mu_{r-1}(V, S_\alpha)$ by means of an r-frame $F^{(r)}$ defined on the regular part V_0 of V near (but off) S_α, as the difference (up to sign) of two classes of $F^{(r)}$ at S_α, the so-called "Schwartz class" and the "virtual class" (Theorems C and D).

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For $r = 1$, the virtual class is an integer, called the virtual index of the vector field. We interpret this index as the Euler-Poincaré invariant $\chi(V') = c^n(V') \frown [V']$ of a "smoothing" V' of V (Theorem E), hence the formula

$$c_0^{FJ}(V) - c_0^{SM}(V) = \chi(V') - \chi(V).$$

When S_α is non-singular, we give more explicit formulas for the computation of $\mu_{r-1}(V, S_\alpha)$ in two cases : for $k = 1$ (hypersurfaces), or when S_α has complex dimension $r - 1$ (Theorem F). Explicit examples appear in [4].

2. Local complete intersections and notation. Let (V, M, E, s) be as above. Since we assume the section s of E to be generically transverse to the zero section, it is regular, and the components of s with repect to a local trivialization of E generate the ideal of (local) holomorphic functions vanishing on V (by [20]). Thus, V is a *local complete intersection in M*. The restriction of E to the regular part V_0 of V may be canonically identified with the normal bundle of V_0 in M. Thus $E|_V$ is an extension to all of V of this normal bundle. We still call it *normal bundle* to V as in the non-singular case. The bundle $E|_V$ depends only on V and not on (E, s).

Let Σ be an analytic subset of V containing the singular part $\mathrm{Sing}(V)$. [In practice, it will be in the sequel the union of $\mathrm{Sing}(V)$ and of the singular part of an r-frame on $V_0 = V - \mathrm{Sing}(V)$]. Denoting by $(S_\alpha)_\alpha$ the set of connected components of Σ, we shall make the following assumption: each S_α is either included in V_0 or is a connected component of $\mathrm{Sing}(V)$, but none of them intersects simultaneously V_0 and $\mathrm{Sing}(V)$.

We will denote by $\{V_i\}$ a Whitney stratification of M compatible with V and Σ. By Łojasiewicz, there is a smooth triangulation (K) of M adapted to V and Σ, and to the given stratification (i.e. having V, the closed stata and Σ as subcomplexes). Let us denote by (K') a first barycentric subdivision of K, and by (D) the cellular decomposition of M, dual to (K), associated to (K'). The i-dimensional skeleton of (D) will be denoted by $(D)^{(i)}$.

For S being one of the S_α, we denote by $\tilde{\mathcal{T}}$ the union of the (D)-cells which intersect S (or equivalently of (D)-cells dual of (K)-simplices in S), and we set $\mathcal{T} = \tilde{\mathcal{T}} \cap V$. We denote by $\tilde{U}$ a neighborhood of S in M containing $\tilde{\mathcal{T}}$, and we set $U = \tilde{U} \cap V$. We shall assume furthermore that $\tilde{U}$ does not intersect the similar neighborhood $\tilde{U}_\alpha$ for other S_α's.

3. Topological definition of Milnor classes.

3-1. Schwartz-MacPherson classes and their localization. One of the original definitions of Chern classes uses the obstruction theory: If V is a complex manifold of dimension n, the Chern class $c^p(V)$ is the obstruction to the existence of a complex r-frame tangent to V on the $2p$-skeleton of a suitable triangulation, where $r = n - p + 1$.

In the case of a stratified singular variety V contained in an m-dimensional complex manifold M, let us write $q = m - r + 1$. We recall that a stratified r-frame on a subset A of M is an r-frame $F^{(r)}$ such that for every $x \in A$, $F^{(r)}(x)$ is tangent to the stratum containing x, in particular, the restriction $F^{(r)}$ of this r-frame to V_0 is tangent to V_0. Among such r-frames there are "radial" r-frames, denoted $F_0^{(r)}$, whose main properties are the following:

(i) for a cell decomposition (D) as above, all vectors of $F_0^{(r)}$ are pointing outwards the neighborhoods $\tilde{\mathcal{T}}$,

(ii) $F_0^{(r)}$ is defined on the $(2q-1)$-dimensional cells of (D) and have isolated singularities inside the $(2q)$-cells,

(iii) the index of $F_0^{(r)}$ in a $(2q)$-cell $\sigma \subset M$ is the same as the index of its restriction to $\sigma \cap V_i \subset V_i$.

The class $c_{SM}^q(V) \in H^{2q}(M, M - V; \mathbf{Z})$ represents the obstruction to extend a radial r-frame $F_0^{(r)}$ to $(D)^{(2q)}$. It does not depend on the choice of $F_0^{(r)}$ as far as it is radial. We refer to [16] and [3] for more precise definitions.

Let us extend the M. H. Schwartz construction to the case where $F^{(r)}$ is a stratified r-frame, not necessarily radial, and defined on V, not necessarily in M.

Also, if $F_1^{(r)}$ and $F_2^{(r)}$ are two (non-singular) r-frames tangent to V_0, defined over the $(D)^{(2q)} \cap (U - S)$, the difference cocycle $d(F_1^{(r)}, F_2^{(r)})$ is well defined in $H^{2p-1}(\partial\mathcal{T})$ (see [18] §33.3): it is the first obstruction for $F_1^{(r)}$ and $F_2^{(r)}$ to be homotopic over $\partial\mathcal{T}$. We denote by $d_S(F_1^{(r)}, F_2^{(r)})$ the image of the class $d(F_1^{(r)}, F_2^{(r)})$ by the composition

$$(*) \qquad H^{2p-1}(\partial\mathcal{T}) \xrightarrow{\delta} H^{2p}(\mathcal{T}, \partial\mathcal{T}) \xrightarrow{\tau} H^{2q}(\tilde{\mathcal{T}}, \partial\tilde{\mathcal{T}}) \xrightarrow{A} H_{2r-2}(S)$$

of the connecting homomorphism δ, the Thom-Gysin homomorphism τ defined by $\langle \tau(c), \sigma\rangle = \langle c, \sigma \cap V\rangle$ for a $2q$-cell σ in (D), and the Alexander isomorphism A (see [2]).

Definition. The "homological Schwartz class" $c^{SM}_{r-1}(F^{(r)}, S)$ of $F^{(r)}$ at S is defined in $H_{2r-2}(S; \mathbf{Z})$ by the formula

$$c^{SM}_{r-1}(F^{(r)}, S) = c^{SM}_{r-1}(S) + d_S(F^{(r)}_0, F^{(r)}).$$

In particular

$$c^{SM}_{r-1}(F^{(r)}_0, S) = c^{SM}_{r-1}(S),$$

for a radial r-frame $F^{(r)}_0$.

Proposition 1. *For two r-frames $F^{(r)}_1$ and $F^{(r)}_2$, we have*:

$$c^{SM}_{r-1}(F^{(r)}_1, S) - c^{SM}_{r-1}(F^{(r)}_2, S) = d_S(F^{(r)}_1, F^{(r)}_2).$$

Theorem C. *Assume that $F^{(r)}$ exists without singularity on $\big(V - \cup_{S_\alpha \subset \Sigma} S_\alpha\big) \cap (D)^{2p}$. Then we get*:

$$c^{SM}_{r-1}(V) = \sum_{S_\alpha \subset V_0} (i_\alpha)_* c^{SM}_{r-1}(F^{(r)}, S_\alpha) + \sum_{S_\alpha \subset \mathrm{Sing}(V)} (i_\alpha)_* c^{SM}_{r-1}(F^{(r)}, S_\alpha),$$

where $i_\alpha : S_\alpha \hookrightarrow V$ denotes the natural inclusion map.

In the situation where S_α is included in V_0, we call $c^{SM}_{r-1}(F^{(r)}, S_\alpha)$ the Poincaré-Hopf class of $F^{(r)}$ at S_α, this terminology being the classical one when r is equal to 1.

3-2. Virtual Chern classes and their localization. We already defined virtual characteristic classes of V as the Chern classes $c^*_{vir}(V) \in H^{2*}(V, \mathbf{Z})$ of the virtual bundle $[TM - E]|_V \in \check{K}(V)$, so they coincide with the usual Chern classes of V when V is non-singular.

Recall the exact sequence

$$0 \to TV_0 \to TM|_{V_0} \to E|_{V_0} \to 0$$

over the regular part V_0 of V. There exists a smooth vector bundle $E' \to V$ and an integer $h \geq 0$, such that $E|_V \oplus E'$ is the trivial bundle θ_h of rank h. Then, $TM|_V \oplus E'$ is an extension of $TV_0 \oplus \theta_h$ to all of V. Every smooth r-frame $F^{(r)}$ tangent to V_0 may be naturally completed as a smooth $(r+h)$-frame of $TM|_{V_0} \oplus E'|_{V_0}$ (denoted by $(F^{(r)}, w^{(h)})$). The virtual class $c^p_{vir}(V)$ may therefore be interpreted in the usual obstruction theory as the first obstruction to the existence of an $(r+h)$-frame of $TM|_V \oplus E'$.

Let $F^{(r)}$ be an r-frame tangent to V_0, defined in particular on the $2p$-skeleton $\partial\mathcal{T} \cap (D)^{2p}$ of $\partial\mathcal{T}$: According to the usual obstruction theory, such $F^{(r)}$ always exist, and the obstruction to extend $(F^{(r)}, w^{(h)})$ without singularity inside of $\mathcal{T} \cap (D)^{2p}$ is a cohomology class $c^p_{vir}(F^{(r)}, S)$ in $H^{2p}(\mathcal{T}, \partial\mathcal{T}; \mathbf{Z}) \cong H^{2p}(V, V - S; \mathbf{Z})$. It does not depend on the choice of E'. Let us denote by $c^{vir}_{r-1}(F^{(r)}, S)$ its image by the composition $A \circ \tau$ (cf (*)).

Proposition 2. *For two r-frames $F^{(r)}_1$ and $F^{(r)}_2$, we have*:

$$c^{vir}_{r-1}(F^{(r)}_1, S) - c^{vir}_{r-1}(F^{(r)}_2, S) = d_S(F^{(r)}_1, F^{(r)}_2).$$

Theorem D. *Assume that $F^{(r)}$ exists without singularity on $\big(V - \cup_{S_\alpha \subset \Sigma} S_\alpha\big) \cap (D)^{2p}$. Then we get*:

$$c^{vir}_{r-1}(V) = \sum_{S_\alpha \subset V_0} (i_\alpha)_* c^{vir}_{r-1}(F^{(r)}, S_\alpha) + \sum_{S_\alpha \subset \mathrm{Sing}(V)} (i_\alpha)_* c^{vir}_{r-1}(F^{(r)}, S_\alpha),$$

where $i_\alpha : S_\alpha \hookrightarrow V$ denotes the natural inclusion map.

For $r = 1$ and $S = \{p\}$ an isolated point, $c^{vir}_0(v, p) \in H_0(\{p\}; \mathbf{Z}) \cong \mathbf{Z}$ is the index defined in [6] if p is a singular point (see §4) and the classical Poincaré-Hopf index if p is a regular point. This justifies the terminology of "Poincaré-Hopf class" when $S \subset V_0$, for every r.

3-3. Milnor classes.

Definition. Let $F^{(r)}$ be an r-frame, as in 3-1. We define

$$\mu_{r-1}(V, S) = (-1)^n \Big[c^{vir}_{r-1}(F^{(r)}, S) - c^{SM}_{r-1}(F^{(r)}, S)\Big],$$

which is in $H_{2r-2}(S; \mathbf{Z})$.

It follows from Propositions 1 and 2 that the homology class $\mu_{r-1}(V, S)$ does not depend on the choice of the stratified r-frame $F^{(r)}$. We shall call it the "homological Milnor class" of V at S.

From Theorems C and D we deduce Theorem A of the introduction. Such a result has been given for local complete intersections with isolated singularities in [15] and [19].

Proposition.

(i) *We have $\mu_{r-1}(V, S) = 0$ for $r > \dim_{\mathbf{C}} S + 1$.*

(ii) *If S is contained in V_0, all the Milnor classes $\mu_{r-1}(V, S)$ vanish.*

4. The case $r = 1$. We already mentioned (Theorem B of the introduction) that $\mu_0(V, S)$ is the classical Milnor number, any time that this number has already been defined. It allows, by the way, to define such a Milnor number in more general situations, for example for local complete intersections of any codimension, S being not necessarily a point. The definition of the Milnor number was given

- for isolated singularities by [11], [7] and [8] in terms of Milnor fiber,

- for hypersurfaces and compact S by [12] in terms of Euler number of some vector bundle of rank m over M.

Observe that none of the methods used in these cases extend to our general situation.

Let us first give a geometric interpretation of the virtual index defined above. We already know ([9], [15]) that if V has an isolated singularity at p and v is a continuous vector field on V, singular only at p, then $c_0^{vir}(v,p)$ equals the GSV-index of v at p, i.e., the Poincaré-Hopf index of an extension of v to a Milnor fibre of V at p.

We have a similar interpretation of the virtual index in the case where the variety V has non-isolated singularities. By Thom's transversality, there exists a C^∞ section s' of E on M which coincides with s on the complement of $\cup_\alpha \tilde{U}_\alpha$ in M, it is transverse to the zero section of E, and it is homotopic to s. We call the zero set V' of s' a C^∞ *smoothing of* V *near* $\mathrm{Sing}(V)$. Denote by $T_{\mathbf{R}}V'$ the (real) tangent bundle of V'; if we consider TM and E as real bundles, we have the exact sequence,

$$0 \to T_{\mathbf{R}}V' \to TM|_{V'} \to E|_{V'} \to 0.$$

Although V' does not have a complex structure, the bundles $TM|_{V'}$ and $E|_{V'}$ are complex vector bundles. Using these, one can use V' to evaluate the virtual index of the restriction v to V of a vector field $\tilde{v}$, defined on $\tilde{U}_\alpha - S_\alpha$, non-singular on $U_\alpha - S_\alpha$, tangent to both V and V', in terms of the Poincaré-Hopf index of the restriction v' of $\tilde{v}$ to V':

Theorem E.

(i) $c_0^{vir}(v, S_\alpha)$ *is equal to the sum of the usual Poincaré -Hopf index of* v' *inside of* $V' \cap \tilde{U}_\alpha$, *independently of* (V', v').

(ii) *In particular,* $c_0^{vir}(V) = \chi(V')$

5. Computations. Let S be a non-singular component of complex dimension ℓ of the singular set $\mathrm{Sing}(V)$. Let H be an $m-\ell$ complex dimensional submanifold of M, transverse to S at a point $x \in S$. Thus x is an isolated singular point of $V \cap H$ in H, and the Milnor number $\mu_0(V \cap H, x)$ does not depend on H as long as it is transverse to S.

Theorem F. *In the above situation*:

(i) *if* $k = 1$, *we have*

$$\begin{aligned}&\mu_{r-1}(V,S)\\ &\quad = (-1)^\ell \mu_0(V \cap H, x)\\ &\qquad \cdot c^{\ell-r+1}([TS - E|_S]) \frown [S],\end{aligned}$$

where $c^{\ell-r+1}([TS-E|_S])$ *denotes the* $\ell - r + 1^{st}$ *Chern class of the virtual bundle* $[TS - E|_S]$.

(ii) *if* $\ell = r-1$ (k *arbitrary*),

$$\mu_{r-1}(V,S) = (-1)^\ell \mu_0(V \cap H, x) \cdot [S].$$

When $k = 1$, a formula conjectured in [21] is proved in [14], giving the "global" Milnor class of V (with arbitrary singularities) as the sum of contributions from each stratum of a stratification of V. The contribution from a non-singular component of $\mathrm{Sing}(V)$ is given as in the formula above.

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