A note on algebraic aspects of boundary feedback control systems of parabolic type

By Takao NAMBU

Department of Applied Mathematics, Faculty of Engineering, Kobe University,
1-1 Rokkodai-cho, Nada-ku, Kobe, Hyogo 657-8501

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1. Introduction. In the study of stabilization of boundary control systems, most fundamental is the static feedback control scheme: Based on a finite number of the observed data (outputs), it is the scheme to feed them back *directly* into the system through the boundary. Let Ω denote a bounded domain of $\mathbb{R}^m$ with the boundary Γ which consists of a finite number of smooth components of $(m-1)$-dimension. The control system studied here is the following initial-boundary value problem:

$$(1)\qquad \begin{aligned} &\frac{\partial u}{\partial t} + \mathcal{L}u = 0 \quad \text{in } (0,\infty)\times\Omega,\\ &\tau u = \sum_{k=1}^{N} \langle u,\, w_k\rangle_\Omega\, h_k \quad \text{on } (0,\infty)\times\Gamma,\\ &u(0,\cdot) = u_0(\cdot) \quad \text{in } \Omega. \end{aligned}$$

Here, $\mathcal{L}$ denotes a uniformly elliptic differential operator of order 2 in Ω defined by

$$\mathcal{L}u = -\sum_{i,j=1}^{m} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j}\Big) + \sum_{i=1}^{m} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u,$$

and $a_{ij}(x) = a_{ji}(x)$ for $1 \leqslant i,\ j \leqslant m,\ \ x \in \overline{\Omega}$. The boundary operator τ associated with $\mathcal{L}$ is either τ_1 of the Dirichlet type or τ_2 of the Robin type:

$$\begin{aligned} \tau_1 u &= u|_\Gamma,\\ \tau_2 u &= \frac{\partial u}{\partial \nu} + \sigma(\xi)u\\ &= \sum_{i,j=1}^{m} a_{ij}(\xi)\nu_i(\xi)\frac{\partial u}{\partial x_j}\Big|_\Gamma + \sigma(\xi)u|_\Gamma, \end{aligned}$$

where $(\nu_1(\xi),\, \dots,\, \nu_m(\xi))$ denotes the unit outer normal at $\xi \in \Gamma$. Necessary regularity on $\overline{\Omega}$ and on Γ of coefficients of $\mathcal{L}$ and τ is assumed tacitly. The inner product and the norm in $L^2(\Omega)$ are denoted by $\langle\cdot,\,\cdot\rangle_\Omega$ and $\|\cdot\|$, respectively. The symbol $\|\cdot\|$ is also used for the $\mathcal{L}(L^2(\Omega))$-norm. In eq. (1), $\langle u,\, w_k\rangle_\Omega$ denote the outputs, where $w_k \in L^2(\Omega)$, and h_k the actuators belonging to $H^{3/2}(\Gamma)$ in the case of the Dirichlet boundary condition, or $H^{1/2}(\Gamma)$ in the Robin boundary condition.

Let us define the linear operators L_i and M_i, $i = 1, 2$ in $L^2(\Omega)$ by

$$\begin{aligned} &L_i u = \mathcal{L}u, \quad u \in \mathcal{D}(L_i),\\ &\mathcal{D}(L_i) = \{u \in H^2(\Omega)\,;\ \tau_i u = 0 \text{ on } \Gamma\} \end{aligned}$$

and

$$\begin{aligned} &M_i u = \mathcal{L}u, \quad u \in \mathcal{D}(M_i),\\ &\mathcal{D}(M_i) = \Big\{u \in H^2(\Omega)\,;\ \tau_i u = \sum_{k=1}^{N}\langle u,\, w_k\rangle_\Omega\, h_k \text{ on } \Gamma\Big\}, \end{aligned}$$

respectively. Henceforth L stands for either L_1 or L_2 when it is distinguished from the context. The same symbolic convention applies to M_i as well as other operators. Eq. (1) is then simply rewritten as the equation in $L^2(\Omega)$:

$$(2)\qquad \frac{du}{dt} + Mu = 0, \quad u(0) = u_0.$$

Given a $\mu > 0$, the problem is to find w_k's and h_k's such that the semigroup $\exp(-tM)$ satisfies the decay estimate

$$(3)\qquad \|e^{-tM}\| \leqslant \text{const}\, e^{-\mu t}, \quad t \geqslant 0.$$

In [4], this estimate was established via the fractional powers L_c^ω, $L_c = L + c$, $c > 0$ and the related fractional calculus. In the case of the Robin boundary condition, for example, we set

$$x(t) = L_{2c}^{-\omega}u(t), \quad \frac{1}{4} < \omega < \frac{1}{2},$$

and, noticing the relation: $\mathcal{D}(L_{2c}^\omega) = H^{2\omega}(\Omega)$ for

$0 \leqslant \omega < 3/4$ [2], turn eq. (2) into

$$\frac{dx}{dt} + L_2 x = \sum_{k=1}^{N} \langle L_{2c}^{\omega} x,\, w_k \rangle_{\Omega} \, L_{2c}^{1-\omega} \psi_k,$$
$$x(0) = L_{2c}^{-\omega} u_0,$$

where $\psi_k \in H^2(\Omega)$ satisfy $(\mathcal{L} + c)\psi_k = 0$, $\tau_2 \psi_k = h_k$, $1 \leqslant k \leqslant N$. The problem is then reduced to that of finding the estimate

$$\left\| \exp\left\{ -t \left(L_2 - \sum_{k=1}^{N} \langle L_{2c}^{\omega} \cdot,\, w_k \rangle_{\Omega} \, L_{2c}^{1-\omega} \psi_k \right) \right\} \right\| \leqslant \text{const}\, e^{-\mu t}, \quad t \geqslant 0.$$

We propose in this note an alternative algebraic approach to the stabilization which requires *no* fractional powers of L_c. The common idea is, however, to turn the problem into another with *no* feedback term on Γ. A merit of the present approach is that the idea is equally applied to a variety of boundary control systems. In fact, the approach via fractional powers requires exact characterization of $\mathcal{D}(L_c^{\omega})$. This seems in general a difficult (but challenging) problem when general elliptic operators with more complicated boundary conditions are studied.

The spectrum $\sigma(L)$ consists only of eigenvalues λ_i, $i \geqslant 1$, lying symmetrically in the interior of a parabola: $\{\lambda = (a\tau^2 - b) + \sqrt{-1}\,\tau;\ \tau \in \mathbb{R}^1\}$, $a > 0$ [1]. They are labelled according to increasing $\operatorname{Re}\lambda_i$. As usual, $P_{\lambda_i} = 1/(2\pi\sqrt{-1}) \int_{|\lambda - \lambda_i| = \varepsilon} (\lambda - L)^{-1}\, d\lambda$ is a projection which maps $L^2(\Omega)$ onto the generalized eigenspace for λ_i, where $\varepsilon > 0$ is small enough. Set $\dim P_{\lambda_i} L^2(\Omega) = m_i\,(< \infty)$, and let $\varphi_{i1}, \dots, \varphi_{im_i}$ be the basis for $P_{\lambda_i} L^2(\Omega)$. As is well known [1], $P_{\lambda_i}^*$ maps $L^2(\Omega)$ onto the generalized eigenspace for $\overline{\lambda_i}$ of L^*, and $\dim P_{\lambda_i}^* L^2(\Omega) = m_i$. The basis for $P_{\lambda_i}^* L^2(\Omega)$ is denoted by $\psi_{i1}, \dots, \psi_{im_i}$.

For a given $\mu > 0$, suppose that

$$\operatorname{Re}\lambda_1 \leqslant \cdots \leqslant \operatorname{Re}\lambda_K \leqslant \mu < \operatorname{Re}\lambda_{K+1}.$$

Set $P = P_{\lambda_1} + \cdots + P_{\lambda_K}$. In view of the expression: $L\varphi_{ij} = \lambda_i \varphi_{ij} + \sum_{k<j} \alpha_{jk}^i \varphi_{ik}$, $1 \leqslant j \leqslant m_i$, the restriction of L onto the invariant subspace $PL^2(\Omega)$ is bounded and similar to the upper triangular matrix Λ, the diagonal elements of which are $\underbrace{\lambda_1, \dots, \lambda_1}_{m_1}, \dots, \underbrace{\lambda_K, \dots, \lambda_K}_{m_K}$. If λ is in $\rho(L_i)$, the boundary value problem:

$$(\lambda - \mathcal{L})\psi_k = 0, \qquad \tau_i \psi_k = h_k,$$
$$1 \leqslant k \leqslant N, \quad i = 1, 2$$

admits a unique solution ψ_k [3] which is denoted by $N_i(\lambda) h_k$,where

$$N_1(\lambda) \in \mathcal{L}(H^{3/2}(\Gamma);\ H^2(\Omega)),$$
$$N_2(\lambda) \in \mathcal{L}(H^{1/2}(\Gamma);\ H^2(\Omega)).$$

The operators $N_i(\lambda)$ are simply rewritten as $N(\lambda)$.

2. Main result. Our first result is

Theorem 2.1.

(i) *The operator M is densely defined. The problem (2) is well posed, and the semigroup e^{-tM} is analytic in $t > 0$.*

(ii) *The adjoint M^* is given by*

$$M_1^* u = \mathcal{L}^* u + \sum_{k=1}^{N} \left\langle \frac{\partial u}{\partial \nu},\, h_k \right\rangle_{\Gamma} w_k,$$
$$u \in \mathcal{D}(M_1^*) = H^2(\Omega) \cap H_0^1(\Omega),$$
$$M_2^* u = \mathcal{L}^* u - \sum_{k=1}^{N} \langle u,\, h_k \rangle_{\Gamma} w_k,$$
$$u \in \mathcal{D}(M_2^*) = \{u \in H^2(\Omega);\ \tau^* u = 0 \text{ on } \Gamma\},$$

where $(\mathcal{L}^*, \tau^*)$ denotes the formal adjoint of $(\mathcal{L}, \tau)$.

For notational convenience, let us introduce the symbol $[u]$ as

$$[u] = \begin{cases} \dfrac{\partial u}{\partial \nu}, & \text{in the case of the Dirichlet boundary condition,} \\ u, & \text{in the case of the Robin boundary condition.} \end{cases}$$

Then M_i^* are simply rewritten as

$$M_i^* u = \mathcal{L}^* u - (-1)^i \sum_{k=1}^{N} \langle [u],\, h_k \rangle_{\Gamma} w_k, \quad i = 1, 2.$$

For a large $c > 0$ with $-c \in \rho(L)$, set $PN(-c)h_k = \sum_{i \leqslant K,\, j} \zeta_{ij}^k \varphi_{ij}$. It is well known -via Green's formula- that there is an $S \times S$ nonsingular matrix A such that $(S = m_1 + \cdots + m_K)$

$$\begin{pmatrix} \zeta_{11}^k \\ \vdots \\ \zeta_{Km_K}^k \end{pmatrix} = A \begin{pmatrix} \langle h_k,\, [\psi_{11}] \rangle_{\Gamma} \\ \vdots \\ \langle h_k,\, [\psi_{Km_K}] \rangle_{\Gamma} \end{pmatrix}.$$

We define the $S \times S$ matrix $\tilde{\Lambda}$ and the $S \times N$ matrix H as

$$\tilde{\Lambda} = A^{-1} \Lambda A, \tag{4}$$

and

$$H = \left(\langle h_k, [\psi_{ij}] \rangle_\Gamma ; \begin{array}{c} k \ \to 1, \dots, N \\ (i,j) \downarrow (1,1), \dots, (K, m_K) \end{array} \right), \tag{5}$$

respectively.

Based on Theorem 2.1, our main result is stated as follows:

Theorem 2.2. *Suppose that $(\tilde{\Lambda}, H)$ is a controllable pair, i.e.,*

$$\operatorname{rank} (H \ \tilde{\Lambda} H \ \tilde{\Lambda}^2 H \ \dots \ \tilde{\Lambda}^{S-1} H) = S. \tag{6}$$

Then there is a set of w_k's $\in P^* L^2(\Omega)$ such that the estimate (3) *holds.*

Outline of the proof. The proof of Theorem 2.1, (i) is almost the same as in [5, Theorem 2.3]: There exists a sector $\bar{\Sigma}_\alpha = \{\lambda - \alpha \in \mathbb{C}; \theta_0 \leqslant |\arg \lambda| \leqslant \pi\}$, $0 < \theta_0 < \pi/2$, $\alpha \in \mathbb{R}^1$, such that

$$\begin{aligned} (\lambda - M)^{-1} f &= (\lambda - L)^{-1} f \\ &+ [N(\lambda) h_1 \dots N(\lambda) h_N] (1 - \Phi(\lambda))^{-1} \cdot \\ &\quad \langle (\lambda - L)^{-1} f, \boldsymbol{w} \rangle_\Omega, \quad \lambda \in \bar{\Sigma}_\alpha, \end{aligned} \tag{7}$$

where $\langle \cdot, \boldsymbol{w} \rangle_\Omega$ denotes the transpose of a vector:

$$(\langle \cdot, w_1 \rangle_\Omega \ \dots \ \langle \cdot, w_N \rangle_\Omega),$$

and

$$\Phi(\lambda) = \left(\langle N(\lambda) h_k, w_j \rangle_\Omega ; \begin{array}{c} k \to 1, \dots, N \\ j \downarrow 1, \dots, N \end{array} \right) \to 0, \quad |\lambda| \to \infty, \ \lambda \in \bar{\Sigma}_\alpha,$$

uniformly. Thus the estimate:

$$\|(\lambda - M)^{-1}\| \leqslant \frac{\text{const}}{1 + |\lambda|}, \quad \lambda \in \bar{\Sigma}_\alpha$$

holds, and e^{-tM} is analytic in $t > 0$.

The expression of the adjoint M_1^* is found in [5, Proposition 2.4], and M_2^* is obtained in almost the same manner as M_1^*.

As to the proof of Theorem 2.2, the main feature is to propose an approach entirely different from and simpler than in [4]. Let us define the operator T by

$$v = Tu = u - \sum_{k=1}^N \langle u, w_k \rangle_\Omega N(-c) h_k. \tag{8}$$

Here the vectors w_j's are to be determined later in relation to $c > 0$ and the associated finite-dimensional stabilization problem (12a). The operator T belongs to $\mathcal{L}(L^2(\Omega)) \cap \mathcal{L}(\mathcal{D}(M); \mathcal{D}(L))$. The bounded inverse T^{-1} exists, and is given by

$$u = T^{-1} v = v + [N(-c) h_1 \dots N(-c) h_N] \cdot (1 - \Phi(-c))^{-1} \langle v, \boldsymbol{w} \rangle_\Omega .$$

Here we have assumed with no loss of generality that $(1 - \Phi(-c))^{-1}$ exists. In fact, consider the case where $\det (1 - \Phi(-c)) = 0$. We then replace w_j's by $(1 + \varepsilon) w_j$'s for a sufficiently small ε. The function $\det (1 - (1 + \varepsilon) \Phi(-c))$ in ε is a polynomial of degree at most N; not a constant; and analytic. Thus $\det (1 - (1 + \varepsilon) \Phi(-c)) \neq 0$ for some small $\varepsilon \neq 0$. As far as ε is small enough, this does not affect the stabilization problem under consideration. The other properties of T are easily examined.

For a solution $u \in \mathcal{D}(M)$ to the problem (2), set

$$v(t) = T u(t), \quad t \geqslant 0. \tag{9}$$

Then $v(t) \in \mathcal{D}(L)$ satisfies the equation

$$\frac{dv}{dt} + T M_c T^{-1} v = cv, \quad t > 0, \quad v(0) = T u_0,$$

where $M_c = M + c$. We calculate as

$$\begin{aligned} T M_c T^{-1} v &= T \mathcal{L}_c \big(v + [N(-c) h_1 \dots N(-c) h_N] \cdot (1 - \Phi(-c))^{-1} \langle v, \boldsymbol{w} \rangle_\Omega \big) \\ &= T \mathcal{L}_c v = T L_c v \\ &= L_c v - \sum_{k=1}^N \langle L_c v, w_k \rangle_\Omega N(-c) h_k. \end{aligned}$$

We assume that w_k's belong to $P^* L^2(\Omega) \subset \mathcal{D}(L^*)$. Then the equation for v is rewritten as

$$\frac{dv}{dt} + Lv - \sum_{k=1}^N \langle v, L_c^* w_k \rangle_\Omega N(-c) h_k = 0, \quad t \geqslant 0, \quad v(0) = T u_0. \tag{10}$$

The problem (10) generates an analytic semigroup. Thus the problem (2) also generates an analytic semigroup $\exp(-tM)$, and

$$\exp(-tM) = T^{-1} \cdot \exp \left\{ -t \left(L - \sum_{k=1}^N \langle \cdot, L_c^* w_k \rangle_\Omega N(-c) h_k \right) \right\} \cdot T, \quad t \geqslant 0. \tag{11}$$

In view of the relation (11), we have to establish a stabilization result for the problem (10). At this stage, the problem is simple since w_k's belong to $P^*L^2(\Omega)$. The restrictions of L onto the invariant subspaces $PL^2(\Omega)$ and $(1-P)L^2(\Omega)\cap\mathcal{D}(L)$ are denoted by L^1 and L^2 respectively. Then, by setting

$$v_1 = Pv, \quad v_2 = (1-P)v,$$

eq. (10) is decomposed into

$$\text{(12}a\text{)}\qquad \frac{dv_1}{dt} + L^1 v_1 - \sum_{k=1}^{N} \langle v_1,\, L_c^* w_k\rangle_\Omega\, PN(-c)h_k = 0,$$

$$\text{(12}b\text{)}\qquad \frac{dv_2}{dt} + L^2 v_2 - \sum_{k=1}^{N} \langle v_1,\, L_c^* w_k\rangle_\Omega\, (1-P)N(-c)h_k = 0.$$

In (12a), replace $L_c^* w_k$ by $y_k = \sum_{i,j\,(i\leqslant K)} y_{ij}^k \psi_{ij}$. Then (12$a$) is equivalent to the equation in $\mathbb{C}^S$:

$$\frac{d\mathrm{v}}{dt} + (\Lambda - Z\overline{Y}\Pi)\mathrm{v} = 0,$$

where

$$Z = \left(\zeta_{ij}^k\,;\, \begin{matrix} k \ \to 1,\dots,N \\ (i,j) \downarrow (1,1),\dots,(K,m_K)\end{matrix}\right) = AH,$$

$$Y = \left(y_{ij}^k\,;\, \begin{matrix} k \ \downarrow 1,\dots,N \\ (i,j) \to (1,1),\dots,(K,m_K)\end{matrix}\right), \quad \text{and}$$

$$\Pi = \left(\langle \varphi_{ij},\, \psi_{pq}\rangle_\Omega\ ;\ \begin{matrix} (i,j) \to (1,1),\dots,(K,m_K) \\ (p,q) \downarrow (1,1),\dots,(K,m_K)\end{matrix}\right).$$

Note that Π is nonsingular and $\langle \varphi_{ij},\, \psi_{pq}\rangle_\Omega = 0$ when $i \neq p$. According to the assumption (6), $(\Lambda,\, Z)$ is a controllable pair, i.e.,

$$\operatorname{rank}(Z\ \Lambda Z\ \Lambda^2 Z\ \dots\ \Lambda^{S-1}Z) = S.$$

Thus the well known pole assignment argument of finite dimension [6] implies that there exists an $N\times S$ matrix Y or w_k's in $P^*L^2(\Omega)$ such that

$$\|e^{-t(\Lambda - Z\overline{Y}\Pi)}\| \leqslant \text{const}\, e^{-\mu t}, \quad t \geqslant 0.$$

By recalling that $\|e^{-tL^2}\| \leqslant \text{const}\, e^{-\mu' t}$, $t \geqslant 0$, $\mu < \mu' < \operatorname{Re}\lambda_{K+1}$, (12$b$) immediately gives the desired estimate for v. Note that μ' cannot be generally replaced by $\operatorname{Re}\lambda_{K+1}$, due to the algebraic multiplicities of the eigenvalues on the vertical line: $\operatorname{Re}\lambda = \operatorname{Re}\lambda_{K+1}$. □

As a concluding remark, another algebraic approach to Theorem 2.2 is possible via Theorem 2.1, (ii). In view of the relation

$$\|e^{-tM}\| = \|(e^{-tM})^*\| = \|e^{-tM^*}\|,$$

the problem is reduced to the one with M^*, and the assumption (6) ensures suitable vectors w_k's in $P^*L^2(\Omega)$.

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