# Remarks on harmonic maps into a cone from a stochastically complete manifold

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**Abstract:** We show that there are no stochastically complete minimal submanifolds immersed in Cartan-Hadamard manifolds which lie in a non-degenerate cone type domain.

**Key words:** Harmonic map; minimal submanifold; stochastically complete; Γ–martingale.

Let $f$ be a smooth map from a complete Riemannian manifold to $\mathbf{R}^n$. Omori showed in [10] that if $f$ is a minimal and isometric immersion and the sectional curvature of $M$ is bounded below, then $f(M)$ can not lie inside any non-degenerate cones in $\mathbf{R}^n$. Non-degenerate cone means such a set as $\{x \in \mathbf{R}^n : (x,\nu) > \delta\|x\|\}$ for some unit vector $\nu$ and constant $\delta > 0$ where $(x,x)$ is standard inner product and $\|x\| = (x,x)^{1/2}$.

Baikoussis and Koufogiorgos showed in [1] that the condition on sectional curvature can be replaced with one that Ricci curvature is bounded from below. Takegoshi([12]) replaced the curvature condition with the growth of the volume of a geodesic ball. He showed that if

$$\liminf_{r\to\infty} \frac{\log V(r)}{r^2} < \infty,$$

the conclusion is valid where $V(r)$ is the volume of a geodesic ball with radius $r$.

All of them showed these results via Omori-Yau maximum principle.

In this note we show that we can replace these assumptions with stochastic completeness of the manifolds and we give a simple proof of a generalized result on this problem without using Omori-Yau maximum principle. We say that a Riemannian manifold $M$ is stochastically complete if

$$\int_M p(t,x,y)dv(y) = 1 \qquad \text{for all} \qquad x \in M,$$

where $p(t,x,y)$ is a heat kernel of $\partial/\partial t - (1/2)\Delta_M$, i.e. Brownian motion on $M$ is conservative. This means that almost all paths of Brownian motions can be extended as their time tends to infinity.
We note that Grigor'yan obtained the following criterion for stochastic completeness ([3]).
Let $M$ be a complete Riemannian manifold and $V(r)$ the volume of a geodesic ball in $M$ of radius $r$. If

$$\int_1^\infty \frac{rdr}{\log V(r)} = \infty,$$

then $M$ is stochastically complete.
Each of the previous three conditions satisfies this condition.

We also define a *cone type domain* of a Riemannian manifold $M$ which is a generalization of Euclidean cone. We call $D$ a cone type domain of $M$ if there exists a nonnegative unbounded function $k$ on $[0,\infty)$ such that $k(d(o,x))$ has a concave majorant on $D$, where $d(o,x)$ is the distance function from a fixed reference point $o$ to $x$ on $M$. If $M = \mathbf{R}^n$, a non-degenerate cone is a cone type domain. Moreover this class includes domains like

$$\{x \in \mathbf{R}^n : (x,\nu) > k(\|x\|)\},$$

where $k$ is an unbounded function on $[0,\infty)$.

Our result is the following.

**Theorem 1.** *Let $N$ be a Cartan-Hadamard manifold and $f$ a harmonic map from $M$ to $N$ satisfying that*

*$(f_*\xi, f_*\xi) \geq c$ for all unit vector fields $\xi$ and some constant $c > 0$. If $M$ is stochastically complete, then $f(M)$ can not lie inside any non-degenerate cone type domains in $N$.*

We immediately have the following corollary to the above theorem.

**Corollary 2.** *No stochastically complete minimal submanifold immersed in a Cartan-Hadamard*

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*manifold $N$ can lie inside a non-degenerate cone type domain in $N$.*

In complex cases the situation becomes slightly simple.

**Corollary 3.** *No stochastically complete complex submanifolds immersed in Cartan-Hadamard manifolds can lie inside a domain which has a form as $D=\{x \in N : \mathrm{Re}\, h(x) \geq k(d(o,x)) > 0\}$ for a holomorphic function $h$ on $N$ and an unbounded function $k$ on $[0,\infty)$.*

If there exists a non-negative exhaustion function $\phi$ on a Kähler manifold $N$ satisfying a condition like $(*)$ in the section 1 below, we can obtain the same result as this replacing $d(o,x)$ and Cartan-Hadamard manifold with $\phi$ and $N$.

We remark that it is impossible to eliminate the assumption of non-degeneracy of cones. In fact we can give an example of non-flat stochastically complete minimal surface in a half space. Moreover we will show the following result in a similar manner to Jorge and Xavier [6].

**Proposition 4.** *There exists a non-flat stochastically complete minimal surface in $\mathbf{R}^3$ between two parallel planes.*

We should remark that Kasue showed in [8] a similar result to our corollary 2 for the case of cylindrical domains of $\mathbf{R}^n$ instead of cones.

He used the fact that if a minimal submanifold is inside a cylindrical domain and $\zeta$ is the life time of a Brownian motion on the submanifold, then $E[\zeta] < \infty$. On the other hand if a minimal submanifold is inside a non-degenerate cone, we may be able only to check that $E[\zeta^{1/2}] < \infty$. Hence we treated some different class of stochastically incomplete manifolds.

We also remark that there are no implication between stochastic completeness and geodesic completeness. Several related results for geodesicaly complete minimal submanifolds are known.
P. Jones ([5]), Jorge and Xavier ([7]) gave examples of bounded complete minimal submanifolds. From our result we know that such surfaces are stochastically incomplete. On the other hand Hoffman and Meeks ([4]) showed, what is called *strong half-space theorem*, that a complete properly immersed minimal surface in $\mathbf{R}^3$ cannot be contained in a half space, except for a plane. As we will see in the last section, a properly immersed minimal surface is always stochastically complete. We can also obtain some similar results on value distribution of harmonic maps under another stochastic condition, which will be given in the last section.

**1. Proof of Theorem 1.** As for stochastic calculus on manifolds and probabilistic notations used here, we can refer to Emery's book [2].

We will show a slightly general result than Theorem 1. Let $M$ and $N$ be a Riemannian manifold. Assume that $N$ admits a nonnegative $C^2-$ class function $\phi(x)$ satisfying the following property.

$$(*) \qquad d\phi \otimes d\phi \leq c_1 \phi(x) g_N, \quad \mathrm{Hess}\,\phi \geq c_2 g_N.$$

We have the following.

**Theorem 5.** *Let $f$ be a harmonic map from $M$ to $N$ satisfying that $g_N(f_*\xi, f_*\xi) \geq c g_M(\xi,\xi)$ for all vector fields $\xi$ and some constant $c > 0$. Let $D = \{x \in N : h(x) \geq k(\phi(x)) > 0\}$ for a continuous function $h$ on $N$ and an unbounded function $k$ on $[0,\infty)$. If $M$ is stochastically complete and $\sup_t E[h(f(X_t))] < \infty$ for a Brownian motion $X_t$ on $M$, then $f(M)$ cannot be included in $D$.*

It is easy to check that if $N$ is a Cartan-Hadamard manifold, then taking $\phi(x) = d(o,x)^2$ makes the condition $(*)$ valid. If a harmonic map $f$ targets $N$ and $h$ is a concave function on $N$, then $h \circ f(x)$ is a superharmonic function on $M$. Then $\sup_t E[h \circ f(X_t)] \leq h \circ f(x_0) < \infty$ for Brownian motion $X_t$ starting from $x_0$. Therefore we make sure that the above theorem implies Theorem 1.

**Proof of Theorem 5.** Suppose that $f(M) \subset D$. Note that the assumption that $g_N(f_*\xi, f_*\xi) \geq c g_M(\xi,\xi)$ for all vector fields $\xi$ and some constant $c > 0$ implies that

$$g_N(df(X), df(X)) \geq c g_M(dX, dX) = cm\, dt$$

becuse if $X_t$ is a Brownian motion on $M$, $g_M(dX,dX) = m dt$ with $m = \dim M$ ([2]).

$\sup_t E[h(f(X_t))] < \infty$ implies that there exists a constant $M > 0$ such that $P(\sup_t k(\phi(f(X_t))) \leq M) > 0$.

Recall Ito's formula of $\Gamma-$martingale ([2]) , that is,

$$\phi(Y_t) - \phi(Y_0) \\ = B\left(\int_0^t d\phi \otimes d\phi(dY, dY)\right) + \frac{1}{2}\int_0^t \mathrm{Hess}\,\phi(dY, dY).$$

By $(*)$

$$d\phi \otimes d\phi(dY, dY)) \leq c_1 \phi(Y_t) g_N(dY, dY)$$

and

$$\text{Hess}\,\phi(dY, dY) \geq c_2 g_N(dY, dY).$$

Law of iterated logarithm of Brownian motion implies that $B(\int_0^t d\phi \otimes d\phi(dY, dY))$ fluctuates within the sides at most of

$$\left(\int_0^t \phi(Y_t) g_N(dY, dY) \log\log \int_0^t \phi(Y_t) g_N(dY, dY)\right)^{1/2},$$

which is bounded by

$$(\sup_t \phi(Y_t)) \left(\int_0^t g_N(dY, dY) \log\log \int_0^t g_N(dY, dY)\right)^{1/2}$$

+ a smaller term, as the clock tends to infinity.
On the other hand

$$\int_0^t \text{Hess}\,\phi(dY, dY) \geq c_2 g_N(dY, dY) \geq const.t.$$

Hence as $t \to \infty$ the right hand side of the Ito's formula for $\phi(Y_t)$ tends to infinity. This contradicts that $\sup_t k(\phi(f(X_t))) < \infty$ with positive probablity.

**2. Proof of Proposition 4.** By Weierstrass representation theorem of minimal surface(cf.[11]), to construct minimal immersion from a unit disc to $\mathbf{R}^3$, we may choose two holomorphic functions on a unit disc satisfying some desirable conditions. In [6] Jorge and Xavier used this strategy. We use the same notation as [6]. Let $K_n$ $(n = 1, 2, \dots)$ be disjoint compact sets in a unit disc such that each $K_n$ is an annulus that a small sector is removed from the right side if $n$ is odd, or from the left side if $n$ is even. Let $Z_t$ be a complex Brownian motion starting from the origin and $\tau = \inf\{t > 0 : |Z_t| \geq 1\}$. We first note that since $Z_t$ converges to $Z_\tau$ in the boundary circle as $t \to \tau$ a.s, $Z$ will cross all but a finite number of the $K_n$ of even $n$'s or all but a finite number of the $K_n$ of odd $n$'s a.s. Let $N_o$ denote this finite random number.

From their argument in [6] we have only to construct a confomal metric satisfying a suitable growth condition to guarantee the stochastic completeness.

Stochastic completeness of the minimal surface implies that

$$\int_0^\tau \sigma(Z_s)^2 ds = \infty, \qquad a.s,$$

where $ds^2 = \sigma(z)^2|dz|^2$ is the induced metric on the unit disc. Now suppose that $ds^2 = \sigma(z)^2|z|^2$ satisfies that $\sigma(z) \geq e^{c_n - 1} > 0$ on $K_n$. Let $d_{hyp}$ be a hyperbolic distance with respect to Poincaré metric on unit disc. We define a hyperbolic Brownian motion $Y$ by

$$Y_{\rho_t} = Z_t \text{ with } \rho_t = \int_0^t \frac{4}{(1 - |Z_s|^2)^2} ds,$$

for $0 \leq t \leq \tau$.
Let $d_{hyp}(o, Y_t) = r_t$. It is well-known that

$$\lim_{t\to\infty} \frac{r_t}{t} = 1 \qquad a.s.$$

Set $S_n = \min\{d_{hyp}(o, x) : x \in K_n\}$ and $T_n = \max\{ d_{hyp}(o, x) : x \in K_n\}$. We show that if $K_n$ and $c_n$ satisfy that for some $0 < \epsilon < 1$ and $\delta > 0$

$$\frac{T_n}{1+\epsilon} - \frac{S_n}{1-\epsilon} \geq \delta \qquad \text{and } c_n \geq \frac{1+\epsilon}{1-\epsilon} S_n,$$

then the minimal surface is stochastically complete.

Fix $0 < \epsilon < 1$ and for this $\epsilon$ there exists a random time $0 < t_0 < \infty$ such that

$$\left|\frac{r_t}{t} - 1\right| < \epsilon \qquad \text{for } t > t_0 \qquad a.s.$$

Set $\tilde{\sigma}(d_{hyp}(o, z)) = \sigma(z)$.

$$\begin{aligned}
&\int_0^\tau \sigma(Z_s)^2 ds \\
&= const. \int_0^\infty \sigma(Y_s)^2 e^{-2r_s} ds \\
&= const. \int_0^{t_0} + \int_{t_0}^\infty \sigma(Y_s)^2 e^{-2r_s} ds \\
&\geq const. \int_{t_0}^\infty \tilde{\sigma}(r_s)^2 e^{-2r_s} ds \\
&\geq const. {\sum}' e^{2c_n - 2} \int_{t_0}^\infty 1_{[S_n, T_n]}(r_s) e^{-2r_s} ds \\
&\geq const. {\sum}' e^{2c_n - 2} \int_{t_0}^\infty 1_{[S_n, T_n]}(r_s) e^{-2r_s} ds \\
&\geq const. {\sum}' e^{2c_n - 2} \\
&\qquad \times \int_{t_0}^\infty 1_{[S_n/(1-\epsilon), T_n/(1+\epsilon)]}(s) e^{-2(1+\epsilon)s} ds,
\end{aligned}$$

where ${\sum}'$ means a summation over all even $n$'s or all odd $n$'s after $N_o$. If the condition on $c_n$ and $K_n$ mentioned above is satisfied, the last sum will be divergent a.s. This completes the proof.

**3. Remarks and similar results.** As mentioned in the introduction, properly immersed minimal surfaces have a stronger property than general minimal sufaces. As for relation between properness of minimal surface and stochastic completeness, we have the following fact.

**Proposition 6.** *A minimal submanifold properly immersed in* $\mathbf{R}^n$ *is stochastically complete.*

**Remark.** 1. Kasue showed a more general result([8]) using analytic method. He gave a condition on mean curvature of the surface and sectional curvature of the ambient manifold for stochastic completeness of the surface.
2. We can show the same conclusion for minimal submanifolds properly immersed in Cartan-Hadamard manifold with lower bounded sectional curvature in the similar way.

*Proof.* Let $f : M \to \mathbf{R}^n$ be a minimal immersion in problem and $X$ Brownian motion on $M$. Direct calculation shows $\Delta_M d(o, f(x))^2 = const.$ Let $\tau_r = \inf\{t > 0 : d(o, f(X_t)) \geq r\}$. Then $E[d(o, f(X_{\tau_r \wedge t}))^2] \leq const.t$. On the other hand properness implies that $\tau_r \uparrow \zeta$ $(r \to \infty)$ a.s. where $\zeta$ is the life time of $X_t$. These facts implies that $d(o, f(X_{\zeta\wedge t})) < \infty$ a.s. for $0 < t < \infty$. This just implies stochastic completeness.

We add another type results in this subject using similar stochastic methods. We can discuss the relation harmonic maps and Liouville property. The details will appear elsewhere.

**Theorem 7.** *Let $f$ be a non-constant harmonic map from a Riemannian manifold $M$ to Cartan-Hadamard manifold $N$ and $D$ a cone type domain as in* Theorem 1 *with $k(x) = x^p$ $(p > 1)$.*

*If $M$ has a Liouville property (i.e. $M$ does not admit any non-constant bounded harmonic functions), then $f(M)$ cannot be included in $D$.*

**Theorem 8.** *Let $f$ be a harmonic map of finite energy from a Riemannian manifold $M$ to Cartan-Hadamard manifold $N$ of lower bounded sectional curvature. If $M$ has a Liouville property and Brownian motion on $M$ is transient, then $f$ is a constant map.*

This is related to the works by Schoen-Yau and Kendall [9]. The former treated the case that $M$ has nonnegative Ricci curvature, $N$ is of non-positive curvature and $f$ is of finite energy. The latter did the cases that $M$ has Liouville property and $f(M)$ is included in a regular geodesic ball. We note that neither of these three results implies the other.

We also remark that there is no implication between Liouville property and stochastic completeness.

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