Dynamics of composite mappings

By Pei-Chu HU*) and Chung-Chun YANG**)***)

(Communicated by Heisuke HIRONAKA, M. J. A., Dec. 14, 1998)

Abstract: In this paper, we will prove some theorems that relate to the dynamics of a composite mapping and its two factors.

Key words: Dynamics; Fatou-Julia theory; composite mappings.

1. Introduction. In this short note, we will generalize some theorems of complex dynamics of one variable for composite functions to several variables cases. In particular, we will prove the following result:

Main theorem. Let f and g be holomorphic self-mappings of degree ≥ 2 on the complex projective space $\boldsymbol{P}^m$ of dimension m. If f, g satisfy $f \circ g = g \circ f$, then $J_{equ}(f) = J_{equ}(g)$.

Here $J_{equ}(f)$ is the Julia set of the mapping f. For the rational function case, that is, $m = 1$, the main theorem gives Theorem 4.2.9 of [3], due to Beardon. For more information on this topic, see, e.g., [1] and [4]. The proof of the main theorem is based on the method used by Beardon and uses a result obtained recently by Ueda [10].

2. Proof of the main theorem. Given a metric space (M, d), denote the set of continuous self-mappings on M by $C(M, M)$. Fix $f \in C(M, M)$. Then there is a maximal open subset so called the Fatou set $F_{equ}(f) = F_{equ}(f, d)$ of M on which the family of iterates $\{f^n\}$ is equicontinuous. Define the Julia set

$$J_{equ}(f) = J_{equ}(f, d) = M - F_{equ}(f, d).$$

It is easy to prove that the sets $F_{equ}(f)$ and $J_{equ}(f)$ are backward invariant if f is an open mapping. Some basic properties of sets $F_{equ}(f)$ and $J_{equ}(f)$ are discussed in Hu-Yang [6] and [7]. By adopting the argument used by Beardon in his proof of Theorem 4.2.9 in [3], the following general result can be obtained:

Theorem 2.1. *If f, $g \in C(M, M)$ are open with $f \circ g = g \circ f$ and satisfy some Lipschitz condition*

$$d(f(x), f(y)) \leq \lambda d(x, y),\ d(g(x), g(y)) \leq \lambda d(x, y),$$

on M, then $f^n(F_{equ}(g)) \subset F_{equ}(g)$ and $g^n(F_{equ}(f) \subset F_{equ}(f)$ for all $n \in \boldsymbol{Z}_+$.

Proof. For any set E, we denote the diameter of E by diam $[E]$ computed using the metric d. Now take $x \in F_{equ}(f)$. By the equicontinuity of $\{f^n\}$ at x, given any positive ε, there is a positive δ such that for all n,

$$\operatorname{diam}[f^n(M(x; \delta))] < \varepsilon/\lambda.$$

As f and g commute we deduce that

$$\begin{aligned}\operatorname{diam}&[f^n \circ g(M(x; \delta))]\\ &= \operatorname{diam}[g \circ f^n(M(x; \delta))]\\ &\leq \lambda \operatorname{diam}[f^n(M(x; \delta))] < \varepsilon.\end{aligned}$$

It follows that $\{f^n\}$ is equicontinuous at $g(x)$, so, in particular, $g(x) \in F_{equ}(f)$. This proves that g, and hence each g^n, maps $F_{equ}(f)$ into itself. We conclude that $g^n : F_{equ}(f) \to F_{equ}(f)$, and so, by symmetry, $f^n : F_{equ}(g) \to F_{equ}(g)$. □

Note that the rational function case is contained implicitly in the proof of Theorem 4.2.9 of [3]. For more information on this topic, see, e.g., [1] and [4]. As an extension of Beardon's result, we prove the following:

Corollary 2.1. *If f, $g \in \mathscr{H}_d$ with $d \geq 2$ satisfying $f \circ g = g \circ f$, then $J(f) = J(g)$, where $\mathscr{H}_d$ is the space of the holomorphic self-mappings on $\boldsymbol{P}^m$ given by homogeneous polynomials of degree d.*

We first note that any C^1 mapping f of a compact Riemannian manifold M satisfies some Lipschitz condition

$$d_M(f(x), f(y)) \leq \lambda d_M(x, y),$$

where d_M is the distance function induced by the

1980 Mathematics Subject Classification (1985 Revision). Primary 58F08, 58F23. Secondary 32H50.

*) Department of Mathematics Shandong University Jinan 250100, Shandong, P. R. China.

**) Department of Mathematics The Hong Kong University of Science & Technology Clear Water Bay, Kowloon, Hong Kong.

***) The work of first author was partially supported by a Post-doctoral Grant of China and second author's was partially supported by the Research Grants Council of Hong Kong Special Administrative Region, China (Project No. HKUST710/96P).

Riemannian metric of M. In fact, we can take

$$\lambda = \sup_{x \in M} \| (df)_x \|.$$

To prove this result, we will need the following facts.

Lemma 2.1. *Let $f: M \to M$ be a distance decreasing mapping i.e., we have*

(1) $d_M(f(x), f(y)) \leq d_M(x, y)$

for all $x, y \in M$. Then $J_{equ}(f) = \emptyset$.

Lemma 2.2 (Ueda [10]). *For any $f \in \mathcal{H}_d$ with $d \geq 2$, the Fatou set $F(f)$ is pseudoconvex, and its connected components are Kobayashi hyperbolic.*

Proof of Corollary 2.1. By Theorem 2.1, we see that $f^n(F_{equ}(g)) \subset F_{equ}(g)$ and $g^n(F_{equ}(f)) \subset F_{equ}(f)$ for all $n \in \mathbf{Z}_+$. Since connected components of $F(g) = F_{equ}(g)$ and $F(f) = F_{equ}(f)$ are Kobayashi hyperbolic, then lemmas above imply that $\{f^n\}$ and $\{g^n\}$ are equicontinuous on $F_{equ}(g)$ and $F_{equ}(f)$, respectively. Therefore we have $F_{equ}(g) \subset F_{equ}(f)$ and $F_{equ}(f) \subset F_{equ}(g)$, respectively, that is, $F_{equ}(g) = F_{equ}(f)$, and hence we obtain $J_{equ}(g) = J_{equ}(f)$. □

3. Dynamics of composite mappings. Let $C(M, N)$ denote the set of continuous mappings from a smooth manifold M into another smooth manifold N. A subset $\mathcal{F}$ of $C(M, N)$ is called *normal*, or a *normal family*, on M iff every sequence of $\mathcal{F}$ contains a subsequence which is either relatively compact in $C(M, N)$ or compactly divergent. We know that for a family $\mathcal{F}$ in $C(M, N)$, we can take the collection $\{U_\alpha\}$ to be the class of all open subsets of M on which $\mathcal{F}$ is normal, this leads to the following general principle.

Theorem 3.1. *Let $\mathcal{F}$ be a family in $C(M, N)$. Then there is a maximal open subset $F(\mathcal{F})$ of M on which $\mathcal{F}$ is normal. In particular, if $f \in C(M, M)$, then there is a maximal open subset $F(f)$ of M on which the family of iterates $\{f^n\}$ is normal.*

The sets $F(\mathcal{F})$ and $F(f)$ in Theorem 3.1 is usually called *Fatou sets* of $\mathcal{F}$ and f respectively. *Julia sets* of $\mathcal{F}$ and f are defined respectively by

$$J(\mathcal{F}) = M - F(\mathcal{F}), J(f) = M - F(f).$$

If $\mathcal{F}$ is finite, we define $J(\mathcal{F}) = \emptyset$. If M is compact, we can prove

$$J_{equ}(f) = J(f).$$

A subset $\mathcal{F}$ of $C(M, N)$ is called uc-normal at $x_0 \in M$ if there exists a neighborhood U of x_0 in M such that $\mathcal{F}|_U = \{f|_U \mid f \in \mathcal{F}\} \subset C(U, N)$ is uc-normal on U, that is, $\mathcal{F}|_U$ is relatively compact in $C(U, N)$. There is a maximal open subset $F_{uc}(\mathcal{F})$ of M on which $\mathcal{F}$ is uc-normal. In particular, if $f \in C(M, N)$; then there is a maximal open subset $F_{uc}(f)$ of M on which the family of iterates $\{f^n\}$ is uc-normal. Thus we obtain a decomposition of the Fatou sets

$$F(\mathcal{F}) = F_{uc}(\mathcal{F}) \cup F_{dc}(\mathcal{F}), F(f) = F_{uc}(f) \cup F_{dc}(f)$$

such that $x \in F_{dc}(\mathcal{F})$ (resp. $F_{dc}(f)$) iff $\mathcal{F}$ (resp. $\{f^n\}$) is normal at x and there exists a sequence of $\mathcal{F}$ (resp. $\{f^n\}$) which is compactly divergent in a neighborhood of x. If U is a component of $F(\mathcal{F})$, we have either $U \subset F_{uc}(\mathcal{F})$ or $U \subset F_{dc}(\mathcal{F})$, i.e.,

$$F_{uc}(\mathcal{F}) \cap F_{dc}(\mathcal{F}) = \emptyset.$$

If N is compact, then $F_{dc}(\mathcal{F}) = \emptyset$ and $F_{uc}(\mathcal{F}) = F(\mathcal{F})$. We also can prove that: If $f \in C(M, M)$ is an open mapping of a smooth manifold M into itself, then $F_{uc}(f)$ and $F_{dc}(f)$ are backward invariant.

Let M be a smooth manifold and take $f, g \in C(M, M)$. Set $h = f \circ g$ and $k = g \circ f$. Note that

(2) $g \circ h^n = k^n \circ g, f \circ k^n = h^n \circ f,$

for all $n \in \mathbf{Z}_+$. We can obtain injective mappings

(3) $g: \mathrm{Fix}(h^n) \to \mathrm{Fix}(k^n)$ and $f: \mathrm{Fix}(k^n) \to \mathrm{Fix}(h^n)$,

for all $n \in \mathbf{Z}^+$, and hence

(4) $g: Per(h) \to Per(k)$ and $f: Per(k) \to Per(h)$.

If f and g are open mappings, by using (2) it is easy to show that:

(5) $g: F_{uc}(h) \to F_{uc}(k)$ and $f: F_{uc}(k) \to F_{uc}(h)$.

Conversely, if $g(x) \in F_{uc}(k)$ for some $x \in M$, then we see $h(x) = f(g(x)) \in F_{uc}(h)$, and hence $x \in F_{uc}(h)$ since $F_{uc}(h)$ is backward invariant. Thus we obtain the following:

Proposition 3.1. *Let f and g be continuous open self-mappings on a smooth manifold M. Then $x \in F_{uc}(f \circ g)$ if and only if $g(x) \in F_{uc}(g \circ f)$.*

Further, we have the following:

Proposition 3.2. *Let f and g be continuous open self-mappings on a smooth manifold M. Let U_0 be a component of $F_{uc}(f \circ g)$ and let V_0 be the component of $F_{uc}(g \circ f)$ containing $g(U_0)$. Then U_0 is wandering if and only if V_0 is wandering.*

Proof. Set $h = f \circ g$ and $k = g \circ f$. Let U_n be the component of $F_{uc}(h)$ containing $h^n(U_0)$ and let V_n be the component of $F_{uc}(k)$ containing $k^n(V_0)$. Then (2) imply

$$g(h^n(U_0)) = k^n(g(U_0)),$$
$$f(k^n(V_0)) = h^n(f(V_0)), n = 1, 2, \ldots,$$

which yield

$$g(U_n) \subset V_n, f(V_n) \subset U_{n+1}.$$

Therefore $U_n = U_m$ implies $V_n = V_m$, and $V_n = V_m$ gives $U_{n+1} = U_{m+1}$. □

If f and g are nonlinear entire functions on $\mathbf{C}$, Baker and Singh [2], Bergweiler and Wang [5], Qiao [9], and Poon-Yang [8] proved independently that the propositions also are true for Fatou sets $F(f \circ g)$ and $F(g \circ f)$. We don't know whether these are true for Fatou sets on high dimensional spaces. If f and g satisfy some Lipschitz condition, we can prove that Proposition 3.1 and 3.2 are true for sets $F_{equ}(f \circ g)$ and $F_{equ}(g \circ f)$. To simplify the notations, the symbols appeared in the following theorem refer to Hu-Yang [6].

Proposition 3.3. *Let $f, g \in \mathrm{Diff}^{\infty}(M, M)$ be two measure preserving mappings and take $p > 0$. Suppose that M is compact, orientable and that f, f^{-1}, g, g^{-1} are orientation preserving. Let μ be the measure induced by a volume form Ω of M. Then $x \in F_{\mu}^{p}(f \circ g)$ if and only if $g(x) \in F_{\mu}^{p}(g \circ f)$.*

Proof. Set $h = f \circ g$ and $k = g \circ f$. Take $x \in F_{\mu}^{p}(h)$. Then there exists a neighborhood U of x such that

$$\lim_{n\to\infty} \| \frac{1}{n} \sum_{j=0}^{n-1} \phi \circ h^j - \bar{\phi} \|_{p,U} = 0,$$

for every $\phi \in C_0(M)$. Since f is orientation preserving and since M is compact, then there is a positive number c such that $0 \leq f^*\Omega/\Omega \leq c$. Thus,

$$\begin{aligned} c^{-1} \| \frac{1}{n} \sum_{j=0}^{n-1} \phi \circ k^j - \bar{\phi}\|_{p,g(U)}^{p} & \\ &\leq \| \frac{1}{n} \sum_{j=0}^{n-1} \phi \circ k^j \circ g - \bar{\phi}\|_{p,U}^{p} \\ &\leq \|\frac{1}{n} \sum_{j=0}^{n-1} \psi \circ h^j - \bar{\psi}\|_{p,U}^{p} \\ &\to 0 \text{ as } n \to \infty, \end{aligned}$$

where $\psi = \phi \circ g$ and $\bar{\psi} = \bar{\phi}$ since g is measure preserving. Therefore $g(x) \in F_{\mu}^{p}(k)$, i.e., $g(F_{\mu}^{p}(h)) \subset F_{\mu}^{p}(k)$. Similarly, we also have $f(F_{\mu}^{p}(k)) \subset F_{\mu}^{p}(h)$.

Conversely, if $g(x) \in F_{\mu}^{p}(k)$ for some $x \in M$, then we see $h(x) = f(g(x)) \in F_{\mu}^{p}(h)$, and hence $x \in F_{\mu}^{p}(h)$ since $F_{\mu}^{p}(h)$ is backward invariant (see Hu-Yang [6]). □

References

[1] I. N. Baker and A. Eremenko: A problem on Julia sets. Ann. Acad. Sci. Fenn., ser. A., **12**, 229–236 (1987).

[2] I. N. Baker and A. Singh: Wandering domains in the iteration of compositions of entire functions. Ann. Acad. Sci. Fenn., ser. **AI20**, pp. 149–153 (1995).

[3] A. F. Beardon: Iteration of Rational Functions. Graduate Texts in Math., **132**, Springer-Verlag. (1991).

[4] A. F. Beardon: Symmetries of Julia sets. Bull. London Math. Soc., **22**, 576–582 (1990).

[5] W. Bergweiler and Y. F. Wang: On the dynamics of composite entire functions. Ark. Mat., **36**, 31–39 (1998).

[6] P. -C. Hu and C. -C. Yang: Invariant sets in dynamics. Bulletin of the Hong Kong Math. Soc., vol. 2, pp. 75–88 (1998).

[7] P. -C. Hu and C. -C. Yang: Fatou-Julia theory in high dimensional spaces. International Workshop on The Value Distribution Theory and Its Applications, Hong Kong. Bull. Hong Kong Math. Soc., **1 (2)**, 273–287 (1997).

[8] K. -K. Poon and C. -C. Yang: Dynamics of composite functions. Proc. Japan Acad., **74A**, 87–89 (1998).

[9] J. -Y. Qiao: Topological complexity of Julia sets. Sci. in China, ser. A, **27**, 775–781 (1997).

[10] T. Ueda: Fatou set in complex dynamics in projective spaces. J. Math. Soc. Japan, **46**, 545–555 (1994).