Special values of zeta functions of the simplest cubic fields and their applications

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1. Preliminaries. In [2], Halbritter and Pohst computed the values of partial zeta functions of totally real cubic fields. In this paper, applying their results to the simplest cubic fields, we explicitly compute some special values of partial zeta-functions of these fields. And as applications, we give a necessary condition for class numbers of the simplest cubic fields to be 1 and construct the simplest cubic fields with class numbers divisible by a given rational integer n.

First we restate the main theorem of [2]. (The meaning of notations such as $\begin{pmatrix} 6 \\ m_1, m_2 \end{pmatrix}$, $B(3, m_1, m_2, 6-(m_1+m_2), (E_\nu B_\rho)^*, 0)$ will be explained in Remarks 1,2 after the statement of the theorem.)

Theorem 1.1 (*Halbritter and Pohst*). *Let K be a totally real cyclic cubic field with discriminant Δ. For $\alpha \in K$ the conjugates are denoted by α' and α'', respectively. Furthermore, for $\alpha \in K$, let $Tr(\alpha) := \alpha + \alpha' + \alpha''$ and $N(\alpha) := \alpha\alpha'\alpha''$. Let $\{\varepsilon_1, \varepsilon_2\}$ be a system of fundamental units of K. Define L by $L := \ln|\varepsilon_1/\varepsilon''_1| \ln|\varepsilon'_2/\varepsilon''_2| - \ln|\varepsilon'_1/\varepsilon''_1| \ln|\varepsilon_2/\varepsilon''_2|$. Let W be an integral ideal of K with basis $\{\omega_1, \omega_2, \omega_3\}$. Let $\rho = \tilde{\omega}_3$ for a dual basis $\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3$ of W subject to*

$$Tr(\omega_i\tilde{\omega}_j) = \delta_{ij} \quad (1 \le i, j \le 3).$$

For $j = 1, 2$, set

$$E_j = \begin{pmatrix} 1 & 1 & 1 \\ \varepsilon_j & \varepsilon'_j & \varepsilon''_j \\ \varepsilon_1\varepsilon_2 & \varepsilon'_1\varepsilon'_2 & \varepsilon''_1\varepsilon''_2 \end{pmatrix}$$

and

$$B_\rho = \begin{pmatrix} \rho\omega_1 & \rho\omega_2 & \rho\omega_3 \\ \rho'\omega'_1 & \rho'\omega'_2 & \rho'\omega'_3 \\ \rho''\omega''_1 & \rho''\omega'''_2 & \rho''\omega''_3 \end{pmatrix}.$$

For $\tau_1, \tau_2 \in K$, $\nu = 1, 2$, set

$$M(2, \nu, \tau_1, \tau_2) := 0$$

if $\det E_\nu = 0$, *otherwise*

$$\begin{aligned} M(2, \nu, \tau_1, \tau_2) :&= -\frac{4\pi^6}{135}\operatorname{sign}(L)(-1)^\nu N(\rho)^2 \frac{\det E_\nu}{|\det(E_\nu B_\rho)|^3} \\ &\cdot \sum_{m_1=0}^{6}\sum_{m_2=0}^{6} \begin{pmatrix} 6 \\ m_1, m_2 \end{pmatrix} \\ &\cdot \{B(3, m_1, m_2, 6-(m_1+m_2), (E_\nu B_\rho)^*, 0) \\ &\cdot \sum_{\varkappa_1=0}^{1}\sum_{\varkappa_2=0}^{1}\sum_{\mu_1=0}^{1}\sum_{\mu_2=0}^{1} \begin{pmatrix} m_1 - 1 \\ 1-(\kappa_1+\kappa_2), 1-(\mu_1+\mu_2) \end{pmatrix} \\ &\cdot \begin{pmatrix} m_2 - 1 \\ \kappa_1, \mu_1 \end{pmatrix} \begin{pmatrix} 5-(m_1+m_2) \\ \kappa_2, \mu_2 \end{pmatrix} \\ &\cdot Tr_{K/\mathbf{Q}}(\tau_1^{\varkappa_1+\varkappa_2}\tau_1'^{\mu_1+\mu_2}\tau_1''^{4-(m_1+\varkappa_1+\varkappa_2+\mu_1+\mu_2)} \\ &\cdot \tau_2^{\varkappa_2}\tau_2'^{\mu_2}\tau_2''^{5-(m_1+m_2+\varkappa_2+\mu_2)})\}, \end{aligned}$$

where $(E_\nu B_\rho)^*$ denotes the transposed matrix of $(E_\nu B_\rho)$, and

$$\begin{aligned} C(2, \nu, \tau_1, \tau_2) :&= -\frac{4\pi^6}{3}\operatorname{sign}(L)(-1)^{\nu+1} \\ &\cdot N(\rho)^2\tilde{B}_4(0)|\det B_\rho|^{-1}\operatorname{sign}(\det E_\nu) \\ &\cdot \{\operatorname{sign}((\tau_1\tau_2 - \tau'_1\tau'_2)(\tau_1 - \tau'_1)) \\ &\qquad + \operatorname{sign}((\tau'_1\tau'_2 - \tau''_1\tau''_2)(\tau'_1 - \tau''_1)) \\ &+ \operatorname{sign}((\tau''_1\tau''_2 - \tau_1\tau_2)(\tau''_1 - \tau_1)) \\ &\qquad + \operatorname{sign}((\tau''_1(\tau_1 - \tau'_1)(\tau'_2 - \tau_2)) \\ &+ \operatorname{sign}((\tau_1(\tau'_1 - \tau''_1)(\tau''_2 - \tau'_2)) \\ &\qquad + \operatorname{sign}((\tau'_1(\tau''_1 - \tau_1)(\tau_2 - \tau''_2)) \\ &+ N(\tau_2)[\operatorname{sign}((\tau''_1(\tau_2 - \tau'_2)(\tau_1\tau_2 - \tau'_1\tau'_2)) \\ &+ \operatorname{sign}((\tau_1(\tau'_2 - \tau''_2)(\tau'_1\tau'_2 - \tau''_1\tau''_2)) \\ &+ \operatorname{sign}((\tau'_1(\tau''_2 - \tau_2)(\tau''_1\tau''_2 - \tau_1\tau_2))]\}. \end{aligned}$$

Define

$$\begin{aligned} \zeta(2, W, \tau_1, \tau_2) :&= M(2, 1, \tau_1, \tau_2) + M(2, 2, \tau_2, \tau_1) \\ &+ C(2, 1, \tau_1, \tau_2) + C(2, 2, \tau_2, \tau_1). \end{aligned}$$

Let $\zeta(s, K_0)$ be the partial zeta function of an absolute ideal class K_0 of K and $W \in K_0^{-1}$. Then we have

$$\zeta(2, K_0) = \frac{1}{2}\operatorname{Norm}(W)^2\zeta(2, W, \varepsilon_1, \varepsilon_2).$$

Remark 1. For $k, l, m \in \boldsymbol{Z}$,

$$\begin{pmatrix} k \\ l, m \end{pmatrix} := \frac{k!}{l!m!(k-(l+m))!},$$

if $k, l, m, k-(l+m) \in \boldsymbol{N} \cup \{0\}$,

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$$\binom{-1}{l,\ m} := (-1)^{l+m}\binom{l+m}{l}, \text{ if } l,\ m \in \boldsymbol{N} \cup \{0\},$$
$$\binom{k}{l,\ m} := 0, \text{ otherwise.}$$

Remark 2. Let $A = (a_{ij})_{n,n}$ be a regular (n, n)-matrix with integral coefficients, $(A_{ij})_{n,n} := (\det A)A^{-1}$. Let

$$\tilde{B}_r(x) := \begin{cases} B_r(x-[x]) & r=0 \text{ or } r\geq 2 \text{ or } r=1 \wedge x \notin \boldsymbol{Z} \\ 0 & r=1 \wedge x \in \boldsymbol{Z} \end{cases},$$

where $B_r(y)$ is defined as usual by $ze^{yz}(e^z - 1)^{-1} = \sum_{r=0}^{\infty} B_r(y) z^r / r!$. Then for $r = (r_1, \ldots, r_n) \in (\boldsymbol{N} \cup \{0\})^n$,

$$B(n, r, A.0) = \sum_{\kappa_1=0}^{|\det A|-1} \cdots \sum_{\kappa_1=0}^{|\det A|-1} \prod_{i=1}^{n} \tilde{B}_{r_i}\Big(\frac{1}{\det A} \sum_{j=1}^{n} A_{ij}\kappa_j\Big).$$

2. Special values of zeta functions of the simplest cubic fields. Let K be the simplest cubic fields defined by the irreducible polynomial over $\boldsymbol{Q}$:

$$f(x) = x^3 + mx^2 - (m+3)x + 1,$$

where m is a rational integer such that $m^2 + 3m + 9$ is square-free. Then the conductor of K is $D = \dfrac{(2m+3)^2 + 27}{4}$. Let α be the negative root of $f(x)$ and $\alpha' = 1/(1-\alpha)$, $\alpha'' = 1 - 1/\alpha$ be its conjugates. Then $\{1, \alpha, \alpha^2\}$ is a basis of K and $\{\alpha, \alpha'\}$ is a system of fundamental units of K. (cf. [5]). Now we have the following lemmas.

Lemma 2.1. *Let q be a factor of $2m+3$. Then (q) is factorized in K with the following form:*

$$(q) = (q, \alpha+1)(q, \alpha-2)(q, \alpha+m+1).$$

Moreover, if $q = 2m+3$ then $(q, \alpha+1) = (\alpha+1)$, $(q, \alpha-2) = (\alpha-2)$ and $(q, \alpha+m+1) = (\alpha+m+1)$ are principal.

Lemma 2.2. *Let $(q, \alpha+1)$ be the ideal of K in Lemma 2.1. Then*

$$(q, \alpha+1) = [q, \alpha+1, \alpha^2+\alpha].$$

From Theorem 1.1, Lemma 2.1 and Lemma 2.2, we have the following theorem.

Theorem 2.3. *Let A be the ideal class of K which contains the ideal $(q, \alpha+1)$ in Lemma 2.1. Then*

$$\begin{aligned}\zeta_K(2, A^{-1}) &= \frac{\pi^6}{D^3}\Big\{\frac{1}{945q^4}m^6 + \frac{1}{105q^4}m^5 \\ &+ \Big(\frac{1}{126q^4} + \frac{1}{20q^2} + \frac{q^2}{3780}\Big)m^4 \\ &+ \Big(-\frac{2}{21q^4} + \frac{3}{10q^2} + \frac{q^2}{630}\Big)m^3 \\ &+ \Big(-\frac{33}{112q^4} + \frac{167}{240q^2} + \frac{139q^2}{15120} + \frac{119}{720}\Big)m^2 \\ &+ \Big(-\frac{183}{560q^4} + \frac{59}{80q^2} + \frac{103q^2}{5040} + \frac{119}{240}\Big)m \\ &+ \Big(-\frac{9}{70q^4} + \frac{3}{10q^2} + \frac{11q^2}{630} + \frac{7}{10}\Big)\Big\}.\end{aligned}$$

In particular, if $q = 1$ or $2m+3$, i.e., A^{-1} is the principal ideal class P of K, then

$$\zeta_K(2, P) = \frac{\pi^6}{D^3}\Big\{\frac{1}{945}m^6 + \frac{1}{105}m^5 + \frac{11}{189}m^4 + \frac{13}{63}m^3 + \frac{544}{945}m^2 + \frac{292}{315}m + \frac{8}{9}\Big\}.$$

Remark 3. Let $W = (q, \alpha+1)$. To prove Theorem 2.3, we need to compute $\zeta(2, W, \alpha, \alpha')$ in Theorem 1.1. But this computation is very long and elementary. So we omit the proof. However we can check our values by comparing them with tables in [2].

3. Applications. Let h_K be the class number of the simplest cubic field K. First we have the following theorem.

Theorem 3.1. *If $h_K = 1$ then $2m+3$ is prime.*

Proof. Suppose that $2m+3 = qr$ ($q \neq 1$ and $r \neq 1$) is a composite integer. If $\zeta_K(2, A^{-1}) = \zeta_K(2, P)$, then

$$(r^2-1)((q^6-q^2)r^4 + (85q^4 - 190q^2 + 105)r^2 + (210q^2 - 210)) = 0.$$

But it is easily seen that this is impossible. Thus $\zeta_K(2, A^{-1}) \neq \zeta_K(2, P)$. So $h_K > 1$ and we have proved the above theorem.

Remark 4. Though all the simplest cubic fields with class number 1 was determined by Lettl [3], it seems interesting to note that Theorem 3.1 can be regarded as a cubic analogue of the following result on quadratic fields [1].

Let $K = \boldsymbol{Q}(\sqrt{4m^2+1})$ for some rational integer m. If $h_K = 1$ then m is prime.

Second we have the following theorem.

Theorem 3.2. *Let n be any positive rational integer. If $2m+3 = a^n$ for some positive rational integer a, then $n \mid h_K$.*

Proof. Let $q = a$. Then from the proof of Theorem 3.1, we have

$$\zeta_K(2, A^{-r}) \neq \zeta_K(2, P) \text{ if } 1 \leq r < n.$$

Thus the order of A^{-1} is n and we have proved the above theorem. □

Remark 5. Imposing some conditions on a

in Theorem 3.2, Uchida [4] and Washington [5] constructed infinitely many cubic fields defined by $f(x)=x^3+mx^2-(m+3)x+1$ with class numbers divisible by a given rational integer n.

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