Three facts of valuation theory

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1. Statement of results. Let k be a field and (R, m) a local integral k-algebra with field of fractions K. We study k-valuations ν of K with a center in R, that is, such that their valuation ring (R_ν, m_ν) contains R and $m_\nu \cap k = (0)$. We denote by Φ the totally ordered group of the valuation and set $\Gamma = \nu(R \setminus \{0\}) \subset \Phi_+ \cup \{0\}$. The valuation determines on R a filtration defined by the ideals

$$P_\phi(R) = \{x \in R / \nu(x) \geq \phi\}$$
$$\text{or } P_\phi^+(R) = \{x \in R / \nu(x) > \phi\},$$

and the associated graded ring introduced by Spivakovsky ([6], see also [4], [7]):

$$\mathrm{gr}_\nu R = \bigoplus_{\phi \in \Gamma} P_\phi / P_\phi^+,$$

which is a Γ (or Φ_+)-graded $(R/m_\nu \cap R)$-algebra. We assume throughout that Φ has finite rational rank $\mathrm{r}(\Phi)$ (and therefore is countable) and finite height (or rank) $\mathrm{h}(\Phi)$ The three facts, extracted from [8], are the following:

1) **A connexion between valuation theory and toric geometry**

Proposition 1.1. *For any specialization* (*see* [10],*vol.*2, *Chap.* VI,§16) *of the valuation* ν *to a valuation* ν_0 *with a center in* R *and such that* $m_{\nu_0} \cap R = m$ *and the residue field extension* $k_R \to k_{\nu_0} = R_{\nu_0}/m_{\nu_0}$ *induced by the inclusion* $R \subset R_{\nu_0}$ *is trivial, the algebra* $\mathrm{gr}_{\nu 0} R$ *is isomorphic to a quotient of a polynomial ring* $k_R[(U_i)_{i \in I}]$ *with coefficients in* k_R *and possibly countably many indeterminates by a binomial ideal, i.e. an ideal with* (*possibly countably many*) *generators of the form* $U^m - \lambda_{mn} U^n$ *where* $U^m = U_1^{m_1} \cdots U_s^{ms}$ *and* $\lambda_{mn} \in k_R^*$. *It means that it is a deformation of a* (*non normal*) *toric variety* (*see* [2]), *possibly of infinite embedding dimension, which is nothing but* $\mathrm{Spec} k_R[t^\Gamma]$, *where* $k_R[t^\Gamma]$ *is the semigroup algebra of* Γ, *obtained by replacing all* λ_{mn} *by* 1.

2) **Structure of valuation semigroup algebras and regularity of** $\mathrm{gr}_\nu R_\nu$

Proposition 1.2. *The graded* k_ν-*algebra* $\mathrm{gr}_\nu R_\nu$ *is a filtering direct limit of termic maps* (*i.e mapping a variable to a term, of the form constant a monomial*) *between polynomial subalgebras in* $\mathrm{r}(\Phi)$ *variables. The semigroup algebra* $k_\nu[t^{\Phi+}]$ *is the direct limit of the corresponding system of toric* (*or monomial*) *maps, obtained by replacing all the constants by* 1.

3) **Noetherianity of** ν**-adic completions**

Proposition 1.3. *Assume that* R *is an analytically irreducible noetherian local ring. If* ν_1 *denotes the valuation of height one with which* ν *is composed, and* $\mathbf{p} = m_{\nu 1} \cap R$ *the center of* ν_1 *on* R, *then the completion* $\hat{R}^\nu$ *of the ring* R *with respect to the topology defined by the* $(P_\phi)_{\phi \in \Phi_+}$-*filtration, is isomorphic as topological ring to a quotient of the* $\mathbf{p}$-*adic completion* $\hat{R}^{\mathbf{p}}$ *of* R; *it is noetherian.*

In particular, if R is excellent, so is $\hat{R}^\nu$ since $\hat{R}^{\mathbf{p}}$ is excellent by ([5]).

2. Ideas of proofs. 1) Since R_ν is a valuation ring, the Φ_+-graded k_ν-algebra $\mathrm{gr}_\nu R_\nu$ has the property that each of its homogeneous components is a 1-dimensional vector space over k_ν. If the residual extension is trivial, the same is true over k_R, and since the k_R-algebra $\mathrm{gr}_\nu R$ is a graded subalgebra of $\mathrm{gr}_\nu R_\nu$, each of its homogeneous components is a k_R-vector space of dimension ≤ 1. By an observation of Korkina ([3], see also [2]), this implies the result: taking a (possibly countable) system of homogeneous generators of the algebra gives a graded surjection $k_R[(U_i)_{i \in I}] \to \mathrm{gr}_\nu R$ once U_i is given the degree of its image. The kernel is generated by homogeneous polynomials, but any two terms of such a polynomial have non zero k_R-proportional images, which shows that the kernel is generated by binomials.

2) Let ν be a valuation of height one, i.e with archimedian value group $\Phi \subset \boldsymbol{R}$ (see [10], Vol. II). Assume first that Φ is generated by m rationally independent positive real numbers $\tau_1, \ldots, \tau_m$. We use the Perron algorithm as expounded in ([9], B. I, p. 861), but with a somewhat different interpretation. The algorithm consists in writing

$\tau_1 = \tau_m^{(1)}, \tau_2 = \tau_1^{(1)} + a_2^{(0)}\tau_m^{(1)}, \ldots, \tau_m = \tau_{m-1}^{(1)} + a_m^{(0)}\tau_m^{(1)}$,
where

$$a_j^{(0)} = [\tau_j/\tau_1], j = 2, \ldots, m,$$

and repeating this operation after replacing $(\tau_1, \ldots, \tau_m)$ by $(\tau_1^{(1)}, \ldots, \tau_m^{(1)})$, and so on. After h steps, one has written

$$\tau_i = A_i^{(h)}\tau_1^{(h)} + \cdots + A_i^{(h+m-1)}\tau_m^{(h)}$$

or, if we denote by w the (weight) vector $(\tau_1, \ldots, \tau_m) \in \boldsymbol{R}^m$ and by $A^{(h)}$ the vector $(A_1^{(h)}, \ldots, A_m^{(h)})$,

$$w = \tau_1^{(h)}A^{(h)} + \tau_2^{(h)}A^{(h+1)} + \cdots + \tau_m^{(h)}A^{(h+m-1)}$$

where the $\tau_j^{(h)}$ are positive, the coefficients $A_i^{(j)}$ are non negative integers, and the matrix of the vectors

$$A^{(h)}, A^{(h+1)}, \ldots, A^{(h+m-1)}$$

has determinant $(-1)^{h(m-1)}$. Moreover, the directions in $\boldsymbol{P}^{m-1}(\boldsymbol{R})$ of the vectors $A^{(h)}$ tend to the direction of w. So we have a sequence of vectors $A^{(h)}$ with positive integral coordinates (becoming longer and longer) whose directions in $\boldsymbol{P}^{m-1}(\boldsymbol{R})$ spiral to the direction of w and such that any consecutive m of them as above form a basis of the integral lattice such that w is contained in the convex cone $\sigma^{(h)} = \langle A^{(h)}, A^{(h+1)}, \ldots, A^{(h+m-1)} \rangle$ which they generate. The convex dual $\check{\sigma}^{(h)}$ of $\sigma^{(h)}$ (see [1], V, 2, p. 149) is contained in the half space $\sum_{j=1}^m a_j\tau_j \geq 0$, the integral points of which form the semigroup Φ_+. For any commutative ring A, the A-algebra of the semigroup $\check{\sigma}^{(h)} \cap \boldsymbol{Z}^m$ is a polynomial algebra $A[x_1^{(h)}, \ldots, x_m^{(h)}]$ (*loc. cit*, VI,2) contained in $A[t^{\Phi_+}]$, and since by assumption there are no integral points on the hyperplane $\sum_{j=1}^m a_j\tau_j = 0$ except the origin, the semigroup Φ_+ is the union of the $\check{\sigma}^{(h)} \cap \boldsymbol{Z}^m$ as $h \to \infty$. This proves that $A[t^{\Phi_+}]$ is the union, or direct limit, of these polynomial subalgebras.

If we now consider a group with one more generator $\tau_{m+1} > 0$ which is rationally dependent on $\tau_1, \ldots, \tau_m$, Zariski shows in ([9], B. I, p. 862) that the new weight vector $w = (\tau_1, \ldots, \tau_m, \tau_{m+1}) \in \boldsymbol{R}^{m+1}$ is contained in a rational simplicial cone $\sigma \subset \boldsymbol{R}^{m+1}$ generated by m integral vectors of the first quadrant forming part of a basis of the integral lattice. Indeed w is contained in a unique rational hyperplane. The dual cone $\check{\sigma} \subset \check{\boldsymbol{R}}^{m+1}$ is the product of an m dimensional strictly convex cone by a 1-dimensional vector space (see [1], V, 2) genetated by an integral vector which is the dual of the rational hyperplane containing w. By refining as above by the Perron algorithm for w inside the linear span $\langle \sigma \rangle$ of σ, starting with the barycentric coordinates of w in σ, we find a sequence of regular simplicial cones $\sigma^{(h)} \subset \sigma$ whose duals $\check{\sigma}^{(h)} \subset \Phi_+$ correspond ([1], VI, Th. 2.12) to algebras of the form $A[x_1^{(h)}, \ldots, x_m^{(h)}, x_{m+1}^{\pm 1}] \subset A[t^{\Phi_+}]$. Note that the map $f : \boldsymbol{Z}^{m+1} \to \boldsymbol{R}$ defined by $(a_1, \ldots, a_{m+1}) \mapsto \sum_{i=1}^{m+1} a_i\tau_i$ is no longer injective; the primitive vector corresponding to the variable x_{m+1} is in the kernel. The $\check{\sigma}^{(h)} \cap \boldsymbol{Z}^{m+1}$ fill up $f^{-1}(\Phi_+ \cup \{0\})$ as $h \to \infty$ since the only rational points of the hyperplane $\sum_{j=1}^{m+1} a_j\tau_j = 0$ are on the dual of the hyperplane containing w, which is contained in all the $\check{\sigma}^{(h)}$. So the direct limit of the images of the maps $A[t^f] : A[x_1^{(h)}, \ldots, x_m^{(h)}, x_{m+1}^{\pm 1}] \to A[t^{\Phi_+}]$ is $A[t^{\Phi_+}]$. But these images are isomorphic to $A[x_1^{(h)}, \ldots, x_m^{(h)}, x_{m+1}^{\pm 1}]/(x_{m+1} - 1)$ so that they are again polynomial rings $A[x_1^{(h)}, \ldots, x_m^{(h)}]$. If we have more generators rationally dependent on $\tau_1, \ldots, \tau_m$, we can repeat the argument after taking as new generators the coordinates of the weight vector with respect to the m primitive vectors of σ. A group $\Phi \subset R$ of finite rational rank is a direct limit of such extensions.

Let now Φ be a totally ordered group of finite height $h > 1$. We have a surjective ordered map $\lambda : \Phi \to \Phi_1$ where Φ_1 is of height $h - 1$, and the kernel Ψ of λ is of height 1. By induction on the height we may assume that Φ_{1+} is the union of sub-semigroups isomorphic to $\boldsymbol{N}^{\mathrm{r}(\Phi_1)}$, and we know from the lemma above that the same is true for Ψ_+.

Let us denote the semigroups that fill Φ_{1+} by F_i, and let $\tilde{F}_i \subset \Phi_+$ be the subsemigroup generated by elements $e_1, \ldots, e_{r_i}$ which lift to Φ_+ the generators of F_i. Similarly let us denote by $G_j \subset \Psi_+$ free semigroups which fill Ψ_+, generated say by $f_1, \ldots, f_{s_j}$. Note that for $\phi \in \Phi_+$, $\psi \in \Psi$, $\phi + \psi \in \Phi_+ \cup \{0\}$, and consider for r_is_j-tuples $n = (n_{s,t}, 1 \leq s \leq r_i, 1 \leq t \leq s_j)$ of non negative integers, the free semigroups $\tilde{F}_i(n) \subset \Phi_+ \cup \{0\}$ generated by $e_1 - \sum_t n_{1t}f_t, \ldots, e_{r_i} - \sum_t n_{r_it}f_t$. The $\tilde{F}_i(n) \oplus G_j$ are of rank equal to the rational rank of Φ and fill up Φ_+. This proves the result.

Corollary 2.1. *The semigroup algebra $k_\nu[t^{\Phi+}]$ is a quotient of a polynomial ring over k_ν in countably many indeterminates $k_\nu[(U_j)_{j\in J}]$ by a binomial ideal with generators of the form*

$$(U_j - U^{m(j)})_{j\in J'}$$

for some $J' \subset J$, *with* $|m(j)| \geq 2$.
The proof above shows that $k_\nu[t^{\Phi+}]$ is a union of polynomial algebras with monomial maps. Each member of a system of homogeneous generators of the k_ν-algebra $k_\nu[t^{\Phi+}]$ must appear in one of the polynomial subalgebras, and we may assume that it appears as one of the generators of that algebra. The only relations between these generators are those corresponding to the toric inclusions $k_\nu[x_1, \ldots, x_m] \to k_\nu[y_1, \ldots, y_m]$, and they are of the announced type.

Corollary 2.2. *Given the ring* R_ν *of a valuation of finite rational rank, the graded algebra* $\mathrm{gr}_\nu R_\nu$ *is a quotient of a polynomial ring* $k_\nu[(U_j)_{j\in J}]$ *in countably many indeterminates over the residue field* k_ν *by a binomial ideal of the form*

$$(U_j - \lambda_j U^{m(j)})_{j\in J'},\ \lambda_j \in k_\nu^*$$

for some $J' \subset J$, *with* $|m(j)| \geq 2$.
it suffices to observe that setting all constants equal to 1 in the binomial relations given by fact 1) for $\mathrm{gr}_\nu R_\nu$ must give $k_\nu[t^{\Phi+}]$.
3) We note that since Φ is countable, the ν-adic topology has a countable basis of neighborhoods for every point. The quotients R/P_ϕ form a projective system since the semigroup Φ_+ is totally ordered. One can then define the ν-adic completion of R as:

$$\hat{R}^\nu = \varprojlim_{\phi\in\Phi_+} R/P_\phi.$$

There is a natural map $R \to \hat{R}^\nu$, which is injective since the filtration is separated. The valuation ν has a canonical extension $\hat{\nu}$ to a valuation of $\hat{R}^\nu$ with values in Φ.
Let us assume that R is a noetherian local ring and that ν is a valuation of height one. Set $\mathbf{p} = m_\nu \cap R$; since R is noetherian, we have $\nu(\mathbf{p}) > 0$, say $\nu(\mathbf{p}) = \phi_0$, and since ν is of height one, its group Φ is archimedian, so that for any $\phi \in \Phi$ there exists an integer $N(\phi)$ such that $N(\phi)\phi_0 \geq \phi$, that is $\mathbf{p}^{N(\phi)} \subset P_\phi$. Then we have a morphism of completions $\hat{R}^{\mathrm{p}} \to \hat{R}^\nu$, where $\hat{R}^{\mathrm{p}}$ is the completion of R for the $\mathbf{p}$-adic topology. Since $\hat{R}^\nu$ is an integral domain, the kernel of this map is a prime ideal $H \subset \hat{R}^{\mathrm{p}}$. By the definition of completions, this kernel is $\cap_{\phi\in\Phi_+} P_\phi \hat{R}^{\mathrm{p}}$. Let us show that the natural injection

$$\hat{R}^{\mathrm{p}}/H \to \hat{R}^\nu$$

is an isomorphism. The ring $\hat{R}^{\mathrm{p}}/H$ is local with maximal ideal $\hat{m} = m\hat{R}^{\mathrm{p}}/H$; it is noetherian, and a Zariski ring with defining ideal $\hat{m}$, as a quotient of a ring having these properties. Moreover, the injection

$$\hat{R}^{\mathrm{p}}/H \subset \hat{R}^\nu$$

implies that the valuation ν extends to $\hat{R}^{\mathrm{p}}/H$ as a valuation $\hat{\nu}$ of height one. The valuation ν is still ≥ 0 on the localization R_{p}, and so we have a morphism of completions $\hat{R}_{\mathrm{p}}^{\mathrm{p}R_{\mathrm{p}}}/H_{\mathrm{p}} \subset \hat{R}_{\mathrm{p}}^\nu$. Let us replace R by R_{p} for a moment, so that $\mathbf{p}$ is the maximal ideal m. By [10], Vol. 2, Appendix 3, Lemma 3 p. 343, the distinct valuation ideals (i.e our $P_\phi(\hat{R}^m/H)$ written without repetition) form a simple infinite descending chain of ideals which are primary for the maximal ideal $\hat{m}$ of $\hat{R}^m/H$ and have intersection zero. We denote them by $\hat{P}_j$. Now we can apply Chevalley's Theorem ([10], Vol. 2, Chap. VIII, § 5, Th. 13, p. 270), which asserts that there exists an integer-valued function $s(n)$ tending to infinity with n and such that for each valuation ideal $\hat{P}_j$, we have $\hat{P}_j \subset \hat{m}^{s(j)}$. This, added to the fact that the $\hat{P}_j$ are primary for $\hat{m}$, proves that in the ring $\hat{R}^m/H$ the $\hat{\nu}$-adic topology coincides with the $\hat{m}$-adic topology, so that it is complete for both, and therefore has to be equal to $\hat{R}^\nu$.

When the center of ν is not necessarily m, this shows that on R we have the inclusions $P_j \subset \mathbf{p}^{(s(j))}$, where $\mathbf{p}^{(k)}$ denotes the symbolic power $\mathbf{p}^{(k)} = \mathbf{p}^k R_{\mathrm{p}} \cap R$. By ([10], Chap VIII, §5, Cor. 5 p. 275), since R is analytically irreducible, the topology defined by the symbolic powers coincides with the $\mathbf{p}$-adic topology, which gives the result in the general case.

It now suffices to note that whenever a valuation ν is composed with a valuation ν_1, the topologies which they define are equivalent.

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