Range Theorems and Inversion Formulas for Radon Transforms on Grassmann Manifolds

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Abstract: In this article, we state some results on the range characterization for Radon transforms on Grassmann manifolds and give the explicit inversion formulas.

Key words: Grassmann manifold; integral geometry; inversion formula; Radon transform; range theorem.

1. Introduction. Let us denote by $\boldsymbol{F}$ the real number field $\boldsymbol{R}$ or the complex number field $\boldsymbol{C}$. Let $Gr(p, n ; \boldsymbol{F})$ be the Grassmann manifold of all p-dimensional subspaces in $\boldsymbol{F}^n$. Then, the Radon transform $R_p^q : C^\infty(Gr(q, n ; \boldsymbol{F})) \to C^\infty(Gr(p, n ; \boldsymbol{F}))$ is defined as follows.

$$R_p^q f(\xi) := \int_{\{\gamma ; \gamma \subset \xi\}} f(\gamma)\, d\mu, \quad \text{if } q < p, \tag{1.1}$$

$$R_p^q f(\xi) := \int_{\{\gamma ; \gamma \supset \xi\}} f(\gamma)\, d\mu, \quad \text{if } p < q, \tag{1.2}$$

for a p-dimensional subspace $\xi \in Gr(p, n ; \boldsymbol{F})$ and for $f \in C^\infty(Gr(q, n ; \boldsymbol{F}))$. Here in (1.1) or in (1.2), $d\mu$ denotes the normalized invariant measure.

Let $s := \min\{q, n-q\} = \operatorname{rank} Gr(q, n ; \boldsymbol{F})$ and $r := \min\{p, n-p\} = \operatorname{rank} Gr(p, n ; \boldsymbol{F})$. If $s < r (\Leftrightarrow \dim Gr(q, n ; \boldsymbol{C}) < \dim Gr(p, n ; \boldsymbol{F}))$, the Radon transform R_p^q is no longer surjective. On the other hand, it is known that R_p^q is injective if $s \le r$. Thus, we arrive at the problems; how to characterize the range of R_p^q and how to reconstruct the inverse image of R_p^q. In fact, for the first problem, Gonzalez [1] shows the existence of the range characterizing operator for R_p^q and for the second problem, Grinberg [3] shows the existence of the inversion formula for R_p^q. However, explicit results for these two problems are still unknown. Therefore, in this article, we give the explicit form of the range characterizing operator and the explicit inversion formula for the Radon transform R_p^q.

The integral geometry on Grassmann manifolds and related subjects will be investigated in our forthcoming paper [9], in which the results in this article will be proved.

2. Complex case. In this section, we deal with the case of complex Grassmann manifolds. The special unitary group $G := SU(n)$ acts on the complex Grassmann manifold $Gr(p, n ; \boldsymbol{C})$ transitively. The stabilizer of the p-dimensional subspace $\boldsymbol{C}\boldsymbol{e}_1 \oplus \cdots \oplus \boldsymbol{C}\boldsymbol{e}_p$ is $K_p := S(U(p) \times U(n-p))$. Then, $Gr(p, n ; \boldsymbol{C})$ can be identified with the compact symmetric space G/K_p.

First, we construct a certain kind of differential operators on the complex Grassmann manifold $Gr(p, n ; \boldsymbol{C})$, which are expressed in terms of determinantal type of differential operators.

Let $\mathfrak{g}$ and $\mathfrak{k}_p$ denote the Lie algebras of G and of K_p, respectively. Then,

$$\mathfrak{g} = \{X \in M_n(\boldsymbol{C}) ; X + X^* = 0, \operatorname{tr} X = 0\},$$

$$\mathfrak{k}_p = \left\{ \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \in \mathfrak{g} ; X_1 \in M_p(\boldsymbol{C}), X_2 \in M_{n-p}(\boldsymbol{C}) \right\}.$$

Let $\mathfrak{g} = \mathfrak{k}_p \oplus \mathfrak{M}_p$ be the Cartan decomposition of the symmetric space G/K_p, where $\mathfrak{M}_p$ is the space of all the matrices X of the form

$$X = \begin{pmatrix} 0 & -Z^* \\ Z & 0 \end{pmatrix} \in \mathfrak{g}, \tag{2.1}$$

$$Z = (z_{i\alpha}) : \text{complex } (n-p) \times p \text{ matrix}$$
$$(1 \le i \le n-p,\ 1 \le \alpha \le p).$$

Let$I = \{i(1), i(2), \ldots, i(d) ; 1 \le i(1) < i(2) < \cdots < i(d) \le n-p\}$ and $A = \{\alpha(1), \alpha(2), \ldots, \alpha(d), ; 1 \le \alpha(1) < \alpha(2) < \cdots < \alpha(d) \le p\}$ be two ordered sets.

For the submatrix Z of X in (2.1) and the above two ordered sets I and A, we define $d \times d$ matrix valued differential operators $\partial Z_{(I,A)}$ and $\partial \bar{Z}_{(I,A)}$ by

$$\partial Z_{(I,A)} := \left(\frac{\partial}{\partial z_{i\alpha}} \right)_{i \in I, \alpha \in A}, \tag{2.2}$$

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$$\partial \bar{Z}_{(I,A)} := \left(\frac{\partial}{\partial \bar{z}_{i\alpha}}\right)_{i \in I, \alpha \in A}.$$

Next, we define d-th order differential operators $L^{(d)}_{(I,A)}$ and $L^{(d)*}_{(I,A)}$ acting on $C^\infty(G)$ by

(2.3) $L^{(d)}_{(I,A)} f(g) := \det \partial Z_{(I,A)} f(g \exp X)\,|_{X=0}$,

(2.4) $L^{(d)*}_{(I,A)} f(g) := (-1)^d \det \partial \bar{Z}_{(I,A)} f(g \exp X)\,|_{X=0}$,

for $f \in C^\infty(G)$. Here X is a matrix of the form (2.1).

Finally, using $L^{(d)}_{(I,A)}$ and $L^{(d)*}_{(I,A)}$, we define a differential operator $\Phi^{(n,p)}_d$ of order $2d$ acting on $C^\infty(G)$ as follows.

$$\text{(2.5)} \quad \Phi^{(n,p)}_d := \sum_{\substack{I \subset \{1,2,\cdots,n-p\} \\ A \subset \{1,2,\cdots,p\} \\ \# I = \# A = d}} L^{(d)*}_{(I,A)} L^{(d)}_{(I,A)},$$

$$(1 \leq d \leq r), \quad \Phi^{(n,p)}_0 := 1.$$

Then, it turns out that the differential operator $\Phi^{(n,p)}_d$ defined by (2.5) is left G-invariant and right K_p-invariant. Therefore, $\Phi^{(n,p)}_d$ is well defined as an invariant differential operator on the symmetric space G/K_p. Then, in the complex case, our range theorem is stated as follows.

Theorem A (*Range theorem-complex case-*). (*I*) *We assume that* $s := \operatorname{rank} G/K_q < r := \operatorname{rank} G/K_p$. *Then, the range* $\operatorname{Im} R^q_p$ *of the Radon transform* R^q_p *is identical with the kernel of* $\Phi^{(n,p)}_{s+1}$, *that is,* $\operatorname{Im} R^q_p = \operatorname{Ker} \Phi^{(n,p)}_{s+1}$. (*II*) *Let* P *be an invariant differential operator on* $Gr(p, n, \boldsymbol{C})$ *of order* $2s+2$ *satisfying the two conditions.* (*a*) $\operatorname{Im} R^q_p \subset \operatorname{Ker} P$. (*b*) *The radial part of* P, $\operatorname{rad}(P)$ *is of the form* $\operatorname{rad}(P) = (-1)^{s+1} 2^{-2(s+1)} S_{s+1}\left(\frac{\partial^2}{\partial t_1^2}, \cdots, \frac{\partial^2}{\partial t_r^2}\right) +$ *lower order terms, where* S_k *denotes the* k-*th elementary symmetric polynomial. Then* P *coincides with* $\Phi^{(n,p)}_{s+1}$.

In the above theorem, $\operatorname{rad}(P)$ is a Weyl group invariant differential operator on the Weyl chamber $\{\exp(t_1 H_1 + \cdots + t_r H_r) K_p \in G/K_p\,;\, 0 < t_r < \cdots < t_1 < \pi/2\}$, where $H_j := \sqrt{-1}(E_{p+j,j} + E_{j,p+j})$ (For the details of the theory of radial parts, see Helgason [5]).

It is well known that any highest weight of the complex Grassmann manifold G/K_p is written of the form $(l_1, \cdots, l_r, 0, \cdots, 0, -l_r, \cdots, -l_1) \in \boldsymbol{R}^n$, where $l_j \in \boldsymbol{Z} (1 \leq j \leq r)$ and $l_1 \geq \cdots \geq l_r \geq 0$. We denote by $V^{(n,p)}(l_1, \cdots, l_r)$ the corresponding irreducible eigenspace of the standard Laplacian on G/K_p. Moreover, we define a homogeneous symmetric polynomial $T_m(t_1, \cdots, t_N)$ of $(t_1, \cdots, t_N)$ by

$$\text{(2.6)} \quad T_m(t_1, \cdots, t_N) := \sum_{1 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_m \leq N} t_{\alpha_1} t_{\alpha_2} \cdots t_{\alpha_m},$$

$$T_0(t_1, \cdots, t_N) := 1.$$

Then, in addition to Theorem A, we have the following eigenvalue formula.

Theorem B (*Eigenvalue formula-complex case-*). *The Schur constant of the invariant differential operator* $\Phi^{(n,p)}_d$ *on the irreducible eigenspace* $V^{(n,p)}(l_1, \cdots, l_r)$ *is given by the formula*

$$\Phi^{(n,p)}_d |_{V^{(n,p)}(l_1,\cdots,l_r)} = \sum_{k=0}^{d} (-1)^{d-k} T_{d-k}(a_d, \cdots, a_r)$$
$$S_k(\chi_1 + a_1, \cdots, \chi_r + a_r),$$

where $\chi_j = l_j(l_j + n + 1 - 2j)$ *and* $a_j = j^2 - (n+1)j + n$.

The above invariant differential operators $\Phi^{(n,p)}_d$ $(1 \leq d \leq r)$ play an important role not only in the range characterization but also in the inversion formula. In fact, the inversion formula for the Radon transform R^q_p is described in terms of the operators $\Phi^{(n,q)}_k$ $(1 \leq k \leq s)$ on the source manifold $Gr(q, n, \boldsymbol{C})$. More precisely, we have the following.

Theorem C (*Inversion formula-complex case-*). *We assume that* $s := \operatorname{rank} G/K_q \leq r := \operatorname{rank} G/K_p$. *Then the Radon transform* R^q_p *on the compact complex Grassmann manifold* $G/K_q = Gr(q, n, \boldsymbol{C})$ *is inverted by the formula*

$$\left\{\prod_{s+1 \leq \alpha \leq s+|p-q|} \sum_{k=0}^{s} \frac{(\alpha-1-k)!(n-\alpha-k)!}{(\alpha-1)!(n-\alpha)!} \Phi^{(n,q)}_k\right\}$$
$$R^p_q R^q_p = I, \quad \text{on } C^\infty(G/K_q).$$

3. Real case. In this section, we deal with the case of real Grassmann manifolds. Therefore, from now on, we denote by G and by K_p the special orthogonal group $SO(n)$ and its subgroup $S(O(p) \times O(n-p))$ respectively. Then the real Grassmann manifold $Gr(p, n; \boldsymbol{R})$ is identified with the compact symmetric space $G/K_p = SO(n)/S(O(p) \times O(n-p))$ in the usual manner. Similarly as in the complex case, we first construct range-characterizing operators.

Let $\mathfrak{g}$ and $\mathfrak{k}_p$ denote the Lie algebras of G and of K_p, respectively. Then

$$\mathfrak{g} = \{X \in M_n(\boldsymbol{R})\,;\, X + {}^t X = 0,\},$$

$$\mathfrak{k}_p = \left\{\begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \in \mathfrak{g}\,;\, X_1 \in M_p(\boldsymbol{R}), X_2 \in M_{n-p}(\boldsymbol{R})\right\}.$$

Let $\mathfrak{g} = \mathfrak{k}_p \oplus \mathfrak{M}_p$ be the corresponding Cartan decomposition. Then $\mathfrak{M}_p$ is the space of all the matrices X of the form

$$\text{(3.1)} \quad X = \begin{pmatrix} 0 & -{}^t Y \\ Y & 0 \end{pmatrix} \in \mathfrak{g},$$

$$Y = (y_{i\alpha}) : \text{real } (n-p) \times p \text{ matrix}$$
$$(1 \leq i \leq n-p,\ 1 \leq \alpha \leq p).$$

Let $I = \{i(1), i(2), \cdots, i(d); 1 \leq i(1) < i(2) < \cdots < i(d) \leq n-p\}$ and $A = \{\alpha(1), \alpha(2), \cdots, \alpha(d), ; 1 \leq \alpha(1) < \alpha(2) < \cdots < \alpha(d) \leq p\}$ be two ordered sets. (Here we assume that $1 \leq d \leq \operatorname{rank} G/K_p = \min\{p, n-p\}$).

For the submatrix Y of X in (3.1) and the above two ordered sets I and A, we define a $d \times d$ matrix valued differential operator $\partial Y_{(I,A)}$ by

(3.2) $$\partial Y_{(I,A)} := \left(\frac{\partial}{\partial y_{i\alpha}}\right)_{i \in I, \alpha \in A}.$$

Next, we define a d-th order differential operator $M^{(d)}_{(I,A)}$ acting on $C^\infty(G)$ by

(3.3) $M^{(d)}_{(I,A)} f(g) := \det \partial Y_{(I,A)} f(g \exp X)|_{X=0}$.

for $f \in C^\infty(G)$. Here X is a matrix of the form (3.1).

Finally, using $M^{(d)}_{(I,A)}$, we define a differential operator $\Phi^{(n,p)}_d$ on $C^\infty(G)$ as follows.

Case I: $2r < n$.

(3.4) $$\Phi^{(n,p)}_d := \sum_{\substack{I \subset \{1,2,\cdots,n-p\} \\ A \subset \{1,2,\cdots,p\} \\ \# I = \# A = d}} (-1)^d 2^{-2d} \{M^{(d)}_{(I,A)}\}^2,$$
$$(1 \leq d \leq r), \quad \Phi^{(n,p)}_0 := 1$$

Case II: $n = 2r$.

(3.5) $$\Phi^{(2r,r)}_d := \sum_{\substack{I \subset \{1,2,\cdots,n-p\} \\ A \subset \{1,2,\cdots,p\} \\ \# I = \# A = d}} (-1)^d 2^{-2d} \{M^{(d)}_{(I,A)}\}^2,$$
$$(1 \leq d \leq r-1),$$
$$\Phi^{(2r,r)}_r := (-\sqrt{-1})^r 2^{-r} M^{(r)}_{(\{1,2,\cdots,r\}, \{1,2,\cdots,r\})},$$
$$\Phi^{(2r,r)}_0 := 1.$$

For the same reason as in the complex case, we see that the above operator $\Phi^{(n,p)}_d$ is well defined as an invariant differential operator on the symmetric space G/K_p. Then, in the real case, our range theorem is given by the following.

Theorem D (*Range theorem-real case-*). (*I*) *We assume that* $s := \operatorname{rank} G/K_q < r := \operatorname{rank} G/K_p$. *Then the range* $\operatorname{Im} R^q_p$ *of the Radon transform* R^q_p *is identical with the kernel of* $\Phi^{(n,p)}_{s+1}$, *that is,* $\operatorname{Im} R^q_p = \operatorname{Ker} \Phi^{(n,p)}_{s+1}$. (*II*) *Moreover, we assume that* $2r < n$. *Let* P *be an invariant differential operator satisfying the same type conditions as* (*a*) *and* (*b*) *in Theorem A. Then,* P *coincides with* $\Phi^{(n,p)}_{s+1}$.

As is well known, if $2r < n$, any highest weight of the real Grassmann manifold G/K_p is written of the form $(2l_1, \cdots, 2l_r, 0, \cdots, 0) \in \boldsymbol{R}^m$ with $l_j \in \boldsymbol{Z}(1 \leq j \leq r)$ and $l_1 \geq \cdots \geq l_r \geq 0$. Here $m = \operatorname{rank} G = [n/2]$. We denote by $V^{(n,p)}(l_1, \cdots, l_r)$ the corresponding irreducible eigenspace of the standard Laplacian on G/K_p. Then the analogous eigenvalue formula holds in the real case.

Theorem E (*Eigenvalue formula-real case-*). *We assume that* $2r < n$. *The Schur constant of the invariant differential operator* $\Phi^{(n,p)}_d$ *on the irreducible eigenspace* $V^{(n,p)}(l_1, \cdots, l_r)$ *is given by the formula*

$$\Phi^{(n,p)}_d|_{V^{(n,p)}(l_1,\cdots,l_r)} = \sum_{k=0}^{d} (-1)^{d-k} T_{d-k}(a_d, \cdots, a_r)$$
$$S_k(\chi_1 + a_1, \cdots, \chi_r + a_r),$$

where $\chi_j = l_j\left(l_j + \frac{n-2_j}{2}\right)$ *and* $a_j = \frac{1}{4}j^2 - \frac{1}{4}nj + \frac{1}{16}n^2$.

However, in the case $2r = n$, the uniqueness of the range-characterizing operator such as Theorem A (*II*) no longer holds. In fact, we have

Proposition F. *If* $2r = n$ *and* $r < 2(s+1) < 2r$, *there exists a one parameter family* $\{\Psi_\nu; \nu \in \boldsymbol{R}\}$ *of invariant differential operators on* $Gr(r, 2r; \boldsymbol{R})$ *such that* $\operatorname{Im} R^q_r = \operatorname{Ker} \Psi_\nu$ *and the radial part of* Ψ_ν, $\operatorname{rad}(\Psi_\nu)$ *is of the form* $\operatorname{rad}(\Psi_\nu) = (-1)^{s+1} 2^{-2(s+1)} S_{s+1}\left(\frac{\partial^2}{\partial t_1^2}, \cdots, \frac{\partial^2}{\partial t_r^2}\right) +$ *lower order terms, for a suitable coordinate system* $(t_1, \cdots, t_r)$ *on a Weyl chamber.*

Finally, we go into the inversion formula for the real Grassmannian case.

Theorem G (*Inversion formula-real case-*). *We assume that* $s := \operatorname{rank} G/K_q \leq r := \operatorname{rank} G/K_p$ *and that* $|p - q|$ *is even. Then, the Radon transform* R^q_p *on the compact real Grassmann manifold* $G/K_q = Gr(q, n, \boldsymbol{R})$ *is inverted by the formula*

$$\left\{\prod_{\substack{s < \alpha \leq s+|p-q| \\ \alpha - s:\text{ even}}} \sum_{k=0}^{s} \frac{2^{2k}(\alpha-2-k)!(n-\alpha-k)!}{(\alpha-2)!(n-\alpha)!} \Phi^{(n,q)}_k\right\}$$
$$R^p_q R^q_p = I, \quad \text{on } C^\infty(G/K_q).$$

We remark that by letting $q = 1$ in the above theorem we obtain the Helgason's inversion formula.

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