Analysis for Stress Intensity Factors in Two Dimensional Elasticity

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(Communicated by Kiyosi ITÔ, M. J. A., May 12, 1997)

Abstract: The aim of the present paper is to give an asymptotic behavior of the displacement field for curved crack case and we give a definition of stress intensity factors for the case. The main result is Theorem 4.

1. Introduction. We study the structure of the displacement field near a tip of a *curved* crack in two dimensional homogeneous and isotropic elasticity, and we restrict ourselves to the analysis for the "stress intensity factors" (S.I.F.'s) in the present research.

The S.I.F.'s are so important that they are regarded as one of the parameters for the criteria of crack growing in fracture mechanics. Solutions to straight crack problems have singularities of order $r^{1/2}$ at the tips of cracks, where r denotes the distance from the tips. Irwin [3] gives an asymptotic expansion of the solution at the tip of a crack to specify the order of singularities. The coefficients of the leading terms are called "stress intensity factors". Refer to Grisvard [2] for a mathematical justification of the expansion in the straight crack case.

Even though the S.I.F.'s are important, mathematical analysis for S.I.F.'s has not established yet for curved crack cases. Most of engineers just believe that solutions have the same structure even for curved cracks as well as straight ones, but we have no mathematical results to guarantee it. We refer Wendland-Stephan [6] concerning a mathematical analysis of singularities in curved crack problems. They show that the "gap of the displacement" on the curved crack has singularity of order $r^{1/2}$ at a tip of the crack, but they do not reach the structure of the displacement field near the tips.

The aim of the present paper is to give asymptotic behavior of the displacement field for a curved crack case and we give a definition of stress intensity factors for the case.

We will show that the structure of the leading terms is independent of given forces and the crack, and the coefficients of the leading terms are worth calling "stress intensity factors". It is a new result that the definition of S.I.F.'s for curved crack problems is given mathematically, and the main result is Theorem 4.

I thank Prof. Ohtsuka and Prof. Iso for their suggestions to the present research.

2. Statement of the problem. Let S be a smooth closed Jordan curve in $\boldsymbol{R}^2$. Let $\Sigma \neq S$ be a connected open subset of S and we denote the end points of Σ by S_1 and S_2. We regard a domain $\Omega_\Sigma := \boldsymbol{R}^2 \setminus \overline{\Sigma}$ as two dimensional isotropic and homogeneous elastic body. We denote by $\mathrm{n} = (n_1, n_2)^T$ the outward unit normal vector on S and $v^{\pm}(x) := \lim_{t\to\pm 0} v(x - t\mathrm{n})$, $x \in S$.

The displacement $\boldsymbol{u} = (u_1, u_2)^T$ is a real valued vector field defined on Ω_Σ. As we restrict our problem to the isotropic and homogeneous case, the strain ε_{ij} and the stress σ_{ij} are given by

$$\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(\partial_j u_i + \partial_i u_j),$$

$$\sigma_{ij}(\boldsymbol{u}) = \lambda \varepsilon_{kk}(\boldsymbol{u})\delta_{ij} + 2\mu\varepsilon_{ij}(\boldsymbol{u}),\ i, j = 1,2,$$

where λ and μ are the Lamé constants and $\partial_j = \partial / \partial x_j$.

Remark 1. *Subscripts take the value* 1,2 *and we use the convention of summation, that is, we omit summation signs over repeated indices. For example,* $\varepsilon_{kk}(\boldsymbol{u}) = \sum_{k=1}^{2} \varepsilon_{kk}(\boldsymbol{u})$ *and* $\partial_j\sigma_{ij}(\boldsymbol{u}) = \sum_{j=1}^{2} \partial_j\sigma_{ij}(\boldsymbol{u})$.

The displacement $\boldsymbol{u}$ is given as a solution to the following problem, which is called the "traction problem".

Problem 1. *Find* $\boldsymbol{u}$ *such that*

(2.1) $$-\partial_j\sigma_{ij}(\boldsymbol{u}) = 0 \text{ in } \Omega_\Sigma,$$

(2.2) $$\sigma_{ij}(\boldsymbol{u})^{\pm} n_j = g_i \text{ on } \Sigma,$$

where $\boldsymbol{g} = (g_1, g_2)^T$ *is a given surface stress on* Σ *called "traction".*

3. Weak formulation. Let us introduce a weighted Sobolev space $W^1(\Omega_\Sigma)$ by

(3.3) $$W^1(\Omega_\Sigma) := \{u \in H^1_{loc}(\Omega_\Sigma) \mid wu \in L^2(\Omega_\Sigma),\ \partial_j u \in L^2(\Omega_\Sigma)\}$$

with a norm

(3.4) $$\| u \|^2_{W^1(\Omega_\Sigma)} := \| wu \|^2_{L^2(\Omega_\Sigma)} + \sum_{j=1}^{2} \| \partial_j u \|^2_{L^2(\Omega_\Sigma)},$$

where w is a weight function $w(x) = (1 + |x|^2 \log^2 |x|)^{-1/2}$ and $|x|$ denotes the Euclidean norm of $x \in \boldsymbol{R}^2$. Functions belonging to $V := W^1(\Omega_\Sigma)^2$ have gaps $[\boldsymbol{u}] := \boldsymbol{u}^+|_\Sigma - \boldsymbol{u}^-|_\Sigma$, and the mapping $\boldsymbol{u} \mapsto [\boldsymbol{u}]$ is continuous linear from V onto $H^{1/2}_{00}(\Sigma)^2$. The space $H^{1/2}_{00}(\Sigma)$ is defined by

$$H^{1/2}_{00}(\Sigma) := \left\{\varphi \in H^{1/2}(\Sigma) \,\middle|\, \int_\Sigma d(x)^{-1}\varphi(x)^2 ds_x < \infty\right\}$$

with a norm $\| \varphi \|^2_{1/2,00} := \| \varphi \|^2_{H^{1/2}(\Sigma)} + \int_\Sigma d(x)^{-1} \varphi(x)^2 ds_x$, where $d(x)$ is a smooth function defined on Σ satisfying

$$\lim_{\Sigma \ni x \to S_j} \frac{d(x)}{|x - S_j|} = d_j \neq 0,$$

and ds_x denotes the measure on Σ with respect to the arcwise parameter. See Lions-Magenes [5] in detail.

For functions belonging to a subspace of V defined by

$$X_0 := \{\boldsymbol{u} \in V \mid \partial_j \sigma_{ij}(\boldsymbol{u}) = 0\},$$

tractions $\sigma_{ij}(\boldsymbol{u})^{\pm} n_j$ are well-defined as continuous linear mappings from X_0 to $\{H^{1/2}_{00}(\Sigma)^2\}'$, and an extended Green's formula

$$0 = \int_{\Omega_\Sigma} - \partial_j \sigma_{ij}(\boldsymbol{u}) v_i dx = \int_{\Omega_\Sigma} \sigma_{ij}(\boldsymbol{u}) \varepsilon_{ij}(\boldsymbol{v}) dx - \langle \sigma_{ij}(\boldsymbol{u})^+ n_j, v_i^+ \rangle + \langle \sigma_{ij}(\boldsymbol{u})^- n_j, v_j^- \rangle$$

holds for $\boldsymbol{u} \in X_0$, $\boldsymbol{v} = (v_1, v_2)^T \in V$.

Remark 2. *The expression* $\langle \cdot, \cdot \rangle$ *is the dual map between* $\{H^{1/2}_{00}(\Sigma)\}'$ *and* $H^{1/2}_{00}(\Sigma)$ *and a function* $\varphi \in L^1(\Sigma)$ *is regarded as an element of* $\{H^{1/2}_{00}(\Sigma)\}'$ *through*

$$\langle \varphi, \phi \rangle = \int_\Sigma \varphi(x) \phi(x) ds_x, \quad \phi \in C^\infty_0(\Sigma).$$

Suppose that $\boldsymbol{u} \in V$ satisfies (2.1) in the sense of distribution and satisfies (2.2) in $\{H^{1/2}_{00}(\Sigma)^2\}'$, then we have

$$\int_{\Omega_\Sigma} \sigma_{ij}(\boldsymbol{u}) \varepsilon_{ij}(\boldsymbol{v}) dx = \langle \boldsymbol{g}, [\boldsymbol{v}] \rangle \text{ for all } \boldsymbol{v} \in V$$

by the Green's formula. But Korn's inequality does not hold for $\boldsymbol{u} \in V$, and we introduce a quotient space, according to Le Roux [4], in order to guarantee uniqueness.

We define an equivalence relation $\boldsymbol{u} \sim \boldsymbol{v}$ by $\boldsymbol{u} - \boldsymbol{v} =$ *constant vector field* and set $\dot{V} := V/\sim$. We denote by $\dot{\boldsymbol{u}}$ the equivalent class containing $\boldsymbol{u} \in V$, and the quotient norm $\| \dot{\boldsymbol{u}} \|_{\dot{V}} := \inf_{\boldsymbol{u} \in \dot{\boldsymbol{u}}} \| \boldsymbol{u} \|_V$ is induced. If we define

$$[\dot{\boldsymbol{u}}] := [\boldsymbol{u}],\ \varepsilon_{ij}(\dot{\boldsymbol{u}}) := \varepsilon_{ij}(\boldsymbol{u}) \text{ and } \sigma_{ij}(\dot{\boldsymbol{u}}) := \sigma_{ij}(\boldsymbol{u}), \quad \text{for } \boldsymbol{u} \in \dot{\boldsymbol{u}},$$

the left hand sides of these expressions are well-defined and we reach the next problem.

Problem 2. *Find* $\dot{\boldsymbol{u}} \in \dot{V}$ *such that*

(3.5) $$\int_{\Omega_\Sigma} \sigma_{ij}(\dot{\boldsymbol{u}}) \varepsilon_{ij}(\dot{\boldsymbol{v}}) dx = \langle g, [\dot{\boldsymbol{v}}] \rangle \textit{ for all } \dot{\boldsymbol{v}} \in \dot{V},$$

for a given $g \in \{H^{1/2}_{00}(\Sigma)^2\}'$.

Now, we have the Korn's inequality

$$\int_{\Omega_\Sigma} \varepsilon_{ij}(\dot{\boldsymbol{u}}) \varepsilon_{ij}(\dot{\boldsymbol{u}}) dx \geq C \| \dot{\boldsymbol{u}} \|^2_{\dot{V}} \text{ for all } \dot{\boldsymbol{u}} \in \dot{V},$$

and hence we conclude that the Problem 2 has a unique solution $\dot{\boldsymbol{u}} \in \dot{V}$ for a given $g \in \{H^{1/2}_{00}(\Sigma)^2\}'$.

Remark 3. *If* $\boldsymbol{u} \in V$ *is a solution to the Problem* 1 *then* $\dot{\boldsymbol{u}}$ *which contains* $\boldsymbol{u}$ *is the solution to the Problem* 2. *Conversely, if* $\dot{\boldsymbol{u}}$ *is the solution to the Problem* 2 *then any representative* $\boldsymbol{u} \in \dot{\boldsymbol{u}}$ *is a solution to the Problem* 1.

4. Representation by the elastic potential. We represent a solution to the traction problem (2.1), (2.2) by the elastic double layer potential defined by

$$W(\varphi)(x) := - \int_\Sigma P(x, y) \varphi(y) ds_y \text{ for } x \notin \Sigma$$

where $P(x, y)$ is a 2×2 tensor with its elements

$$P_{ij}(x, y) = \frac{1}{2\pi} \frac{\partial}{\partial n_y} \log \frac{1}{|x - y|} \delta_{ij} + \frac{\mu}{2\pi(\lambda + \mu)} \frac{\partial}{\partial s_y} \log \frac{1}{|x - y|} \varepsilon_{ij} + \frac{\lambda + \mu}{2\pi(\lambda + 2\mu)} \frac{\partial}{\partial s_y} \frac{(x_i - y_i)(x_k - y_k)}{|x - y|^2} \varepsilon_{kj},$$

$\varphi(x) = (\varphi_1(x), \varphi_2(x))^T \in L^2(\Sigma)^2$ and the symbol ε_{ij} denotes the alternating symbol

$$(\varepsilon_{ij}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Lemma 1. *For any* $\varphi \in H^{1/2}_{00}(\Sigma)^2$, *we have*:

(i) $W(\varphi) \in W^1(\Omega_\Sigma)^2 \cap C^\infty(\Omega_\Sigma)^2$ *and* $\partial_j \sigma_{ij}(W(\varphi)) = 0$ *in* Ω_Σ,

(ii) $\sigma_{ij}(W(\varphi))^+ n_j = \sigma_{ij}(W(\varphi))^- n_j$ *in* $\{H^{1/2}_{00}(\Sigma)^2\}'$.

We denote by $\sigma_{ij}(W(\varphi)) n_j$ the same values $\sigma_{ij}(W(\varphi))^+ n_j$ and $\sigma_{ij}(W(\varphi))^- n_j$ by Lemma 1 (ii).

If we find $\varphi \in H^{1/2}_{00}(\Sigma)^2$ satisfying

(4.6) $$\sigma_{ij}(W(\varphi)) n_j = g_i,$$

then $W(\varphi)$ is regarded as a solution to Problem 1 by this lemma. Indeed, we find the density function satisfying (4.6) as the following manner.

Define a bilinear form on $H_{00}^{1/2}(\Sigma)^2$ by

$$b(\varphi, \psi) := \langle \sigma_{ij}(W(\varphi))n_j, \psi_i \rangle,$$

then we find that the bilinear form is symmetric, continuous and coercive on $H_{00}^{1/2}(\Sigma)^2$, and we have the following theorem.

Theorem 2. *Suppose* $g \in \{H_{00}^{1/2}(\Sigma)^2\}'$, *then there uniquely exists* $\varphi \in H_{00}^{1/2}(\Sigma)^2$ *such that*

$$b(\varphi, \psi) = \langle \boldsymbol{g}, \psi \rangle \text{ for all } \psi \in H_{00}^{1/2}(\Sigma)^2.$$

5. Structure of the solution. In order to define S.I.F.'s, we specify the structure of the displacement field by calculating behaviors of $W(\varphi)$ in the neighborhoods $B_\delta(S_j) := \{x \in \boldsymbol{R}^2 \mid |x - S_j| < \delta\}$ $(j = 1,2)$, where the constant $\delta > 0$ is so small that $B_\delta(S_1) \cap B_\delta(S_2) = \emptyset$.

We introduce local coordinates $(x_1^{(j)}, x_2^{(j)})$ with their origins S_j and we assume the coordinates satisfy that

(i) the $x_1^{(j)}$-axis is tangential to the crack Σ at the end point S_j,

(ii) the crack Σ lies in $\{(x_1^{(j)}, x_2^{(j)}) \mid x_1^{(j)} < 0\}$ locally near S_j,

with the orthogonal basis $\boldsymbol{e}_1^{(j)}, \boldsymbol{e}_2^{(j)} (j = 1,2)$.

If we assume suitable regularity of the given traction $\boldsymbol{g}$, we find singularity of the density function by using the pseudo-differential operator theory. We summarize the procedure as follows: We regard the operator mapping φ to $\sigma_{ij}(W(\varphi))n_j$ as a pseudo-differential operator which has a principal symbol

$$\frac{(\lambda + \mu)\lambda}{(\lambda + 2\mu)} \begin{pmatrix} |\xi| & 0 \\ 0 & |\xi| \end{pmatrix},$$

and we obtain the singularity of φ by the Wiener-Hoph method shown in Èskin [1].

Theorem 3. *Suppose* $g \in H^{1/2}(\Sigma)^2$, *then the solution* $\varphi \in H_{00}^{1/2}(\Sigma)^2$ *to* (4.6) *has the following form*:

$$\text{(5.7)} \qquad \varphi = \varphi_S + \varphi_R,\ \varphi_R \in H_{00}^{3/2}(\Sigma)^2,$$

$$\text{(5.8)} \begin{pmatrix} \varphi_S \cdot \boldsymbol{e}_1^{(j)} \\ \varphi_S \cdot \boldsymbol{e}_2^{(j)} \end{pmatrix} = \begin{pmatrix} k_2^{(j)} \\ k_1^{(j)} \end{pmatrix} r_j^{1/2} \text{ on } \Sigma \cap B_\delta(S_j) \quad j = 1,2,$$

where the distance between $x \in \Sigma$ *and the end point* S_j *is denoted by* r_j, $(j = 1,2)$.

Remark 4. *Wendland and Stephan show similar result in* [6], *but their expression of the coefficients of singular term has no meaning as S.I.F.'s.*

We obtain asymptotic behavior of the displacement field near crack tips from the solution to the Problem 1 represented by the elastic double layer potential $W(\varphi)$.

Theorem 4. *Suppose* $g \in H^{1/2}(\Sigma)^2$, *then the solution* $\boldsymbol{u} = W(\varphi)$ *has the following form*: $\boldsymbol{u} = \boldsymbol{u}_S + \boldsymbol{u}_R$, $\boldsymbol{u}_R|_{\Omega_\Sigma \cap B_\delta(S_j)} \in H^2(\Omega_\Sigma \cap B_\delta(S_j))^2$,

$$\begin{pmatrix} \boldsymbol{u}_S \cdot \boldsymbol{e}_1^{(j)} \\ \boldsymbol{u}_S \cdot \boldsymbol{e}_2^{(j)} \end{pmatrix} = \frac{k_1^{(j)} r_j^{1/2}}{\kappa + 1} \begin{pmatrix} \cos\left(\frac{\theta_j}{2}\right)(\kappa - \cos\theta_j) \\ \sin\left(\frac{\theta_j}{2}\right)(\kappa - \cos\theta_j) \end{pmatrix} + \frac{k_2^{(j)} r_j^{1/2}}{\kappa + 1} \begin{pmatrix} \sin\left(\frac{\theta_j}{2}\right)(\kappa + 2 + \cos\theta_j) \\ -\cos\left(\frac{\theta_j}{2}\right)(\kappa - 2 + \cos\theta_j) \end{pmatrix},$$

$\kappa = (\lambda + 3\mu)/(\lambda + \mu)$,

$-\pi + \bar{\theta}_j(r_j) < \theta < \pi + \bar{\theta}_j(r_j)$, *in* $\Omega_\Sigma \cap B_\delta(S_j)$, $j = 1,2$. *Here,* (r_j, θ_j) *denotes the polar coordinate with its center* S_j *for each* j, *and* $\bar{\theta}_j(r_j)$ *is defined as the angle between the negative part of the* $x_1^{(j)}$*-axis and a point* $x \in \Sigma$.

The structure of the displacement shown in the last theorem is similar to that of the straight crack problem except for the range of the angle. Therefore it is natural that we define S.I.F.'s for the curved crack problem through the coefficients of the singular terms which are shown in the last theorem. Indeed, if we define $K_I^{(j)}$ and $K_{II}^{(j)}$ by

$$\frac{K_I^{(j)}}{\sqrt{2\pi}\mu} = \frac{k_1^{(j)}}{\kappa + 1} \text{ and } \frac{K_{II}^{(j)}}{\sqrt{2\pi}\mu} = \frac{k_2^{(j)}}{\kappa + 1}, \quad j = 1,2$$

using the coefficients $k_1^{(j)}$ and $k_2^{(j)}$ given in the Theorem 3, the expression of the displacement field corresponds to the straight crack case.

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